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Mathematical Reviews

Vol. 20, No. 6 JUNE, 1959 Reviews 3772-4474

LOGIC AND FOUNDATIONS

See also 4454.

3772:

Schmidt, Jürgen. Einige Prinzipien der doppelten Induktion. Math.-Phys. Semesterber. 6 (1958), 137–147.

Der Aufsatz beschäftigt sich mit den induktiven (besser: rekursiven) Methoden des Beweises eines Satzes von der Form: die der Aussagenform $\mathfrak{H}(x, y)$ entsprechende Punktepaarmenge $\mu(=\{\langle x, y\rangle\})$ ist gleich $\mathcal{N}\times\mathcal{N}$, wobei $\mathcal{N}$ die Menge aller natürlichen Zahlen ist. Es ist evident, dass hierzu der Beweis (wenn er nicht versagt) des einfachen Induktionsschemas (1.1) $\mathfrak{H}(a, 0)$, (1.2) $\mathfrak{H}(a, b)\rightarrow\mathfrak{H}(a, fb)$ bei beliebigem aber festem a, wobei fb der Nachfolger von b ist, ausreicht. Das folgende im Grunde immer noch einfachere, nicht wirklich doppelte Induktionsverfahren: (2.1) $\mathfrak{H}(a, 0)$, (2.2) $(\forall x)\mathfrak{H}(x, b)\rightarrow\mathfrak{H}(a, fb)$ reicht auch dazu aus. Der Verfasser zeigt nunmehr u.a., dass die folgenden, wirklich doppelten Induktionsverfahren ebenfalls ausreichen: (3.1) $\mathfrak{H}(a, 0)$, (3.2) $(\forall x)\mathfrak{H}(x, b)\rightarrow\mathfrak{H}(0, fb)$, (3.3) $(\forall x)\mathfrak{H}(x, b)\wedge\mathfrak{H}(a, fb)\rightarrow\mathfrak{H}(fa, fb)$; (4.1) $\mathfrak{H}(a, 0)$, (4.2) $\mathfrak{H}(0, b)$, (4.3) $\mathfrak{H}(a, b)\wedge\mathfrak{H}(fa, b)\wedge\mathfrak{H}(a, fb)\rightarrow\mathfrak{H}(fa, fb)$; (6.1) $\mathfrak{H}(a, 0)$, (6.2) $\mathfrak{H}(a, b)\wedge\mathfrak{H}(b, a)\rightarrow\mathfrak{H}(a, fb)$ ("symmetrische doppelte Induktion"). *B. Germansky* (Berlin)

3773:

Markov, A. A. On the inversion complexity of function systems. Dokl. Akad. Nauk SSSR (N.S.) 116 (1957), 917–919. (Russian)

Let B_n be the free Boolean algebra with n indeterminates, the operations being & (meet), $\vee$ (union), and $\neg$ (negation). Given a system of m formulas $P_1, \cdots, P_m$ from B_n, the inversion complexity of this system is the number of distinct formulas of the form $\neg A$ which are subformulas of at least one P_i. Let an n-place Boolean function be a function $f(x_1, \cdots, x_m)$ which assigns a value from the range $\{0, 1\}$ to each n-tuple of values from the same range; every formula of B_n defines such a function and, conversely, every such function is defined by various such formulas. The inversion complexity of a system of m functions $f_1, \cdots, f_m$ is then defined as the minimal inversion complexity of a system of formulas $P_1, \cdots, P_m$ such that P_i defines f_i. [This concept for $m=1$ was mentioned by E. N. Gilbert in J. Math. Phys. 33 (1954), 57–67; MR 15, 1009.] The author's principal theorem relates this concept for the case $m=1$ to another concept which may be called the alternating length. A sequence $A_1, \cdots, A_r$ of $\{0, 1\}$-valued n-tuples is called an alternating chain for a Boolean function f just when (a) each A_{i+1} is obtained from A_i by changing one or more zeros into ones, $f(A_1)=1$, and the $f(A_i)$ are alternately 1 and 0. The alternating length of f is then 0 if $f\equiv 0$ and otherwise is the maximal value of r for such a chain. If this r is called Alt (f) and the inversion complexity is Im (f), the author's result is Inv $(f)=$pd $(D(\text{Alt }(f)))$, where $D(r)=[\log_2 r]+1$, and pd is the predecessor function of primitive recursive arithmetic. The proof is, of course, not given; but it is said to depend on some lemmas which are too complex for statement here.

H. B. Curry (University Park, Pa.)

3774:

Zhuravlev, Yu. I. On the separability of subsets of the vertices of an n-dimensional unit cube. Dokl. Akad. Nauk SSSR (N.S.) 113 (1957), 264–267. (Russian)

An assignment of values 0, 1 to each of the variables x_i in a sequence $x_1, x_2, \cdots, x_n$ corresponds in an obvious way to a vertex of the n-dimensional unit cube. This paper is concerned with functions which assign one of the values 0, 1, to such vertices; if the assignment is made for all vertices the function will be here called total, otherwise it will be called partial. Given a partial function $F(x_1, \cdots, x_n)$, the problem of this paper is to find a total function f which is an extension of F, such that f can be expressed in a disjunctive normal form containing a minimal number of letters. (Such an f separates the sets of vertices for which $F=0$ from those for which $F=1$.) The author gives a complete solution of this problem and some related questions. The solution is rather a common sense one, involving a large number of steps; the reviewer is not able to judge its practicality.

H. B. Curry (University Park, Pa.)

3775:

✱Zhuravlev, Iu. I. On the separability of subsets of the vertices of an n-dimensional unit cube. Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 6 pp.

A translation of #3774 reviewed above.

3776:

Shen, Yu-Ting. The basic calculus. Acta Math. Sinica 7 (1957), 132–143. (Chinese. English summary)

The basic calculus may be conceived as a part of the intersection of Johanson's Minimalkalkül [Compositio Math. 4 (1936), 119–136] and Lewis's modal calculus S_4 [C. I. Lewis and C. H. Langford, Symbolic Logic, Century, New York-London, 1932]. It uses two primitive inference-schemas, with 14 axiom-schemas.

From the author's summary

3777:

Rose, Alan. An alternative formalisation of Sobociński's three-valued implicational propositional calculus. Z. Math. Logik. Grundlagen Math. 2 (1956), 166–172.

Sobociński gave an axiomatization of the 3-valued propositional calculus with two designated truth-values and having as primitives the two functors C and N. {This, incidentally, is a very special case of the problem solved by Angelo Margaris in "A Problem of Rosser and Turquette in many-valued logic" [Thesis, Cornell University, 1956].} The author here axiomatizes Sobociński's system using C as the only primitive and modus ponens and substitution as the only rules of inference.

L. N. Gál (New Haven, Conn.)

3778:

Schmidt, H. Arnold. Die Gesamtheit der Idempotenten implikativen Modalitätenstrukturen. Arch. Math. Logik Grundlagenforsch. 3 (1957), 29–49.

Here the investigations of the author's previous paper [same Arch. 2 (1956), 33–54; MR 19, 724], are continued using the previous paper as groundwork. Again it is assumed that the basic modalities 'possible' and 'necessary' are idempotent. Here it is shown that in this case there are only finitely many implicative modal structures. They and their interrelations are explicitly exhibited.

L. N. Gál (New Haven, Conn.)

3779:

Friedman, Joyce. Some results in Church's restricted recursive arithmetic. J. Symb. Logic 22 (1957), 337–342.

Restricted recursive arithmetic was introduced by A. Church [same J. 20 (1955), 286–287] in a review of Berkeley, Sci. Monthly 78 (1954), 232–243. This system provides for the definition of functions by simultaneous recursion. The author of the present paper shows that any such recursion can be replaced by a single set of recursions of a simple form. The decision problem for the system is solved. That is, an effective procedure is provided for determining whether a given formula Z (given the recursion equivalences which define its nonprimitive functional constants) is valid. The author also shows that the system of restricted recursive arithmetic is complete.

E. J. Cogan (Bronxville, N.Y.)

3780:

Dekker, J. C. E.; and Myhill, J. Some theorems on classes of recursively enumerable sets. Trans. Amer. Math. Soc. 89 (1958), 25–59.

The authors extend the results of Rice [same Trans. 74 (1953), 358–366; MR 14, 713] and start a classification of recursively enumerable (r.e.) classes which parallels the classification of recursively enumerable sets initiated by Post [Bull. Amer. Math. Soc. 50 (1944), 284–316; MR 6, 29]. Let F be the class of all r. e. sets; $\omega_1, \omega_2, \omega_3, \cdots$ be a sequence in which all r. e. sets are effectively generated; and S be a subclass of F. Define $\theta S=\{n|\omega_n \in S\}$. The authors, through this definition, reduce treatment of notions concerning classes of r. e. sets to the treatment of sets of positive integers. Lower bounds are determined for the degrees of unsolvability of certain classes S, and the degree of unsolvability of the class Q of all finite sets is found. The notions of separable classes and productive classes are introduced and investigated. A topology is imposed on the class F by specifying certain subclasses as open, which provides a new proof that F has no nontrivial decidable subclasses. This is accomplished by showing that every effective operation on F is continuous, an operation Φ being called effective if there is a recursive function f such that $\Phi(\omega_n)=\omega_{f(n)}$. Errata: p. 26, l.–8: "decidable" for "undecidable"; p. 31, l.–13: $(\exists\forall)'$ for $('\exists\forall)$.

E. J. Cogan (Bronxville, N.Y.)

3781:

Kalmár, László. On Church's hypothesis as foundation for studies related to the so-called unsolvable mathematical problems. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 7 (1957), 19–38. (Hungarian)

3782:

Kokoszyńska, Maria. On "good" and "bad" induction. Studia Logica 5 (1957), 43–70. (Polish. Russian and English summaries)

After a terminological discussion the considerations are based on the axioms for a two-argument probability function $P(x, y)$ as presented by Helmer and Oppenheim [J. Symb. Logic 10 (1945), 25–60; MR 7, 45; p. 29]. The author proposes the following definitions of a non-relativised (I) notion of correctness of inductive inference and two versions (II.1 and II.2) of a relativised notion of correctness of inductive inference: (I) An inference with a premise (conjunction of premises) f and conclusion h is correct if and only if $P(h, f\cdot w)>P(h, w)$, where w is the set of presuppositions; (II.1) An inference with a premise f and conclusion h is more correct than an inference with a premise f' and conclusion h' if and only if $P(h, f\cdot w)/P(h, w)>P(h', f'\cdot w)/P(h', w)$; (II.2) An inference etc. (as in II.1) if and only if $P(h, f\cdot w)>P(h', f'\cdot w)$. Parallel distinctions of incorrect inference follow.

H. Hiż (Philadelphia, Pa.)

3783:

Teodorescu, N. Fondements d'une théorie générale des grandeurs et des opérations. I. La notion générale de grandeur. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) 1 (49) (1957), 355–368.

Der Verfasser will den allgemeinen Begriff der Grösse definieren, und er tut das im wesentlichen durch die Forderung, dass ein jedes der Elemente einer Menge von Grössen (z.B. der Menge aller reellen Zahlen) identifizierbar sein soll, d.h. eine Eigenschaft haben soll, die ihm und nur ihm zukommt. Dies dürfte aber nicht genügen, und seine Definition ist zu allgemein. Denn, wie Haupt [Einführung in die Algebra, Band I, Akademische Verlag, Leipzig, 1929; S. 9] angedeutet hat, muss eine Menge von Grössen darüber hinaus eine (wenigstens teilweise) Ordnung ihrer Elemente aufweisen. Der Verfasser kann nämlich nicht willkürlich den Begriff der Grösse, was sonst erlaubt wäre, definieren, da dieser Begriff schon bekannt ist, und viele Verwirklichungen gefunden hat. Der Verasser scheint nicht die umfangreiche Literatur über diesen Gegenstand berücksichtigt zu haben.

Ferner ist an dem Aufsatz zu beanstanden, dass Verf. eine Idee Haupts [loc. cit. S. 7] übernimmt, ohne deren Urheber zu nennen (obgleich er Haupt an anderer Stelle zitiert). Er nennt nämlich Gleicheit, was Haupt mit gutem Recht als Identität ("feinste" Gleichheit) bezeichnet. Dadurch allein fasst Verf. natürlich noch keine neue Idee.

Die Untersuchung der bestehenden mathematischen Grundbegriffe (wie Mehrheit, Totalität, Klasse, Menge, Grösse usf.) bereitet bekanntlich fast unüberwindliche Schwierigkeiten. Verf. hat diese Schwierigkeiten, was den Begriff der Grösse betrifft, wohl unterschätzt.

B. Germansky (Berlin)

3784:

Mostowski, Andrzej. The present state of investigations on the foundations of mathematics. Advancement in Math. 3 (1957), 127–163. (Chinese)

A translation of an article that has appeared in Polish and Russian; the Russian version [Uspehi Mat. Nauk N.S. 9 (1954), no. 3(61), 3–38] was reviewed in MR 16, 552.

3785:

Greniewski, Henryk. On Mill's method of concomitant variations. Studia Logica 5 (1957), 109–127. (Polish. Russian and English summaries)

The notion of a relatively isolated system is defined by a long list of postulates. It resembles in many respects some current literature about automata. A relatively isolated system has inputs, outputs, calendar, repertory. Most of the traditional methods of induction are analyzable in terms of relatively isolated systems with

calendar composed of time segments, input and output being 0, 1, or $\{0, 1\}$ exclusively. The method of concomitant variations requires systems with infinite sets of 0 and 1 as inputs and outputs. The paper is an improvement over a previous book by Greniewski [Elements of inductive logic, 1955 (Polish)]. *H. Hiž* (Philadelphia, Pa.)

3786:

Ianov, Yu. I. On matrix schemes. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 283–286. (Russian)

Let s range over the 2^m logical m-tuples (i.e., m-tuples $\langle p_1, \cdots, p_m \rangle$, where each p_i is either 0 or 1). Let M be a matrix of $n(n+1)$ logical functions (i.e., with values 0 or 1) $\alpha_{ij}(s)$, $i=0, 1, \cdots, n$; $j=1, 2, \cdots, n$. Given a sequence $S=s_0, s_1, \cdots$ of m-tuples, a sequence $i_0, i_1, \cdots$ of integers will be called a value of S just when it is determined as follows: one takes $i_0=0$; having determined i_k, one searches in M for a j such that $\alpha_{ij}(s)=1$ for $i=i_k$ and $s=s_k$, and takes one such j for i_{k+1}; if there is no such j the sequence terminates with i_k. Various restrictions on the change from s_k to s_{k+1} in relation to i_k are considered, and a sequence S is said to be admissible if it determines at least one value subject to the restrictions. The matrix M with conditions is called a matrix scheme just when it determines at most one value (possibly void) for every admissible S. The author states theorems giving a necessary and sufficient condition that a matrix be a matrix scheme, that two matrix schemes be equivalent (in the sense that they determine the same value), and that a matrix scheme admit certain transformations. Various connections with program systems, citing an earlier paper [same Dokl. **113** (1957), 39–42; MR **19**, 985], are discussed, and it is stated that equivalence of program schemes can be reduced to that of matrix schemes. More general sorts of equivalence, relating to cases where the integers i determine certain operators A_i, between which there may be relations, are also mentioned; it is stated that they are related to the associative systems of Markov [The theory of algorithms, Trudy Mat. Inst. Steklov. no. 42, Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **17**, 1038], and in general lead to unsolvable problems; but the method of matrix schemes leads to a positive solution in a certain simple case. {There is a misprint in Theorem 1; the theorem is false as stated, but becomes valid if A^* is replaced by A^{**}; this agrees with the other theorems.} *H. B. Curry* (University Park, Pa.)

SET THEORY

See also 3804, 4272.

3787:

Farah, Edison. On the countable union of countable sets. Ciência e Cultura **10** (1958), 86. (Portuguese and English)

Let $S \subseteq I\omega = \{0, 1, 2, \cdots\}$; for each $n \in S$ let A_n be a countable set and f_n a biunique mapping of A_n into $I\omega$. For each $x \in \bigcup A_n = A$ $(n \in S)$ let p be the least ordinal satisfying $x \in A_p$; then f is a biunique mapping of A into $I\omega$. *Đ. Kurepa* (Zagreb)

3788:

Iséki, Kiyoshi. On a theorem of W. Sierpiński and S. Ruziewicz. Proc. Japan Acad. **34** (1958), 353–354.

Restatements of known theorems. *L. Gillman* (Princeton, N.J.)

3789:

Rothberger, F. Exemple effectif d'un ensemble transfiniment non-projectif. Canad. J. Math. **10** (1958), 554–560.

L'auteur construit un ensemble E qui n'appartient à aucune classe de rang dénombrable d'ensembles projectifs. (En partant par exemple de l'espace J des nombres irrationnels, on définit les P_1-ensembles ou A-ensembles comme projections sur J des G_δ-ensembles de J^2, $P_2=CP_1$, $P_3=PP_2$, etc., P_ω sont des unions des ω-suites des ensembles de classe $<\omega$, $P_{\omega+1}=PCP_\omega$, etc. [Cf. W. Sierpiński, Les ensembles projectifs et analytiques, Gauthier-Villars, Paris, 1950; MR **14**, 627.]) Soit $n \to r_n$ une application biunivoque de l'ensemble N des entiers positifs sur l'ensemble Q des nombres rationnels; à chaque $t=\sum_1^\infty 2^{-n}t_n$, $t_n \in \{0, 1\}$ soit $T=\{r_n; t_n=1, n \in N\}$; on ordonne T par $<$. Soit π_3 une homéomorphie de J^3 sur J; soit $x \to (\pi_{r_1}(x), \pi_{r_2}(x), \cdots)$ une homéomorphie de J sur J^ω. On définit par induction relativement à T la fonction propositionnelle $\phi(t, x, y)$ $(0 \leqq t < 1,\ x \in J,\ y \in J)$ de la manière suivante: $\phi(0, x, y) . \equiv . (\exists r)\pi_r(y) < x < r$; $\phi(t, x, y) . \equiv . (\exists z)(z')(\exists r)\phi(t_r, \pi_3(x, z, z'), \pi_r(y))$ si T est non vide et bien ordonné; t_r désigne le segment de t jusqu'à r; enfin $\phi(t, x, y) . \equiv . 0=0$ si T n'est pas bien ordonné. Alors l'ensemble $\hat{t}$, $\hat{x}$, $\hat{y}\phi(t, x, y)$ est l'ensemble désiré E. *Đ. Kurepa* (Zagreb)

3790:

Perić, Veselin. Généralisation d'un théorème sur les relations. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. **13** (1958), 39–40. (Serbo-Croatian summary)

The author proves the following theorem [conjectured by Sierpiński and proved for $n=2$ by S. Piccard, Fund. Math. **28** (1937), 197–202]. Let E be a non-denumerable set, and let R_n be an n-ary relation on E such that for each $x^1 \in E$ there exist at most finitely many sequences $x_i^2, x_i^3, \cdots, x_i^n$ of elements of E with $(x^1, x_i^2, \cdots, x_i^n) \in R_n$. Then there exists an $E_1 \subset E$ of the same cardinal as E such that if $y^1, y^2, \cdots, y^n$ is any set of n distinct elements of E_1, then $(y^1, y^2, \cdots, y^n) \notin R_n$. The proof proceeds by induction on n. *G. Sabidussi* (New Orleans, La.)

3791:

Wang, Shuh-tang; and Wang, Keh-shien. On some equations of ordinal numbers. Advancement in Math. **3** (1957), 646–649. (Chinese. English summary)

The equations

$$\xi^n = \eta^{n+1}+1, \quad \xi^n m = \eta^{n+1}+1,$$

where n and m are natural numbers with $n>1$, $m \neq 2$, have no solutions in transfinite ordinal numbers ξ, η [generalization of Sierpiński, Fund. Math. **43** (1956), 1–2; MR **17**, 1190]. Also, the equations $\alpha m + \beta n = \beta n + \alpha m$ and $\alpha + \beta = \beta + \alpha$ are equivalent [generalization of Sierpiński, ibid., 139–140; MR **17**, 1190].

From the authors' summary

COMBINATORIAL ANALYSIS

See also 3912.

3792:

Nunnink, H. J. C. A. Permutations of n elements, in which no two successive elements stand in natural order. Nieuw Tijdschr. Wisk. **45** (1957/58), 254–257. (Dutch)

The problem deals with circular permutations, and can

be formulated as follows. Consider permutations P of the set of all n residue classes mod n, with the property that $P(b+1)-P(b)\not\equiv 1 \pmod{n}$ for every residue class b. It is shown that the number of such permutations is $(-1)^n n + n \cdot n! \sum_{k=0}^{n-1} (-1)^k/(k!(n-k))$, and that this is $\sim n! e^{-1}$ if $n \to \infty$. *N. G. de Bruijn* (Amsterdam)

3793:

Nilov, G. N. Calculation of the 2nd and 3rd characteristics of a matrix *B*. Kabardin. Gos. Ped. Inst. Uč. Zap. 12 (1957), 24–27. (Russian)

The functions $f(2, n, m)$, $f(3, n, m)$ are calculated. Here $f(r, n, m)$ is the number of combinations of r elements of an n by m matrix chosen from distinct rows under the condition that no two of the r elements lie on the main right-to-left diagonal nor on any one of the right-to-left diagonals below the main one.

3794:

Fine, N. J. Classes of periodic sequences. Illinois J. Math. 2 (1958), 285–302.

Let q symbols $1, 2, \cdots, q$ be given and consider all sequences of length n; place two sequences in the same equivalence class if they can be obtained from one another by merely renaming the symbols or by starting the one sequence at a different point. For example, if $q=3$, $n=4$, the sequences (1123), (3312), and (3123) are all equivalent. From these finite sequences, we may build up infinite sequences by repetition; the primitive period of such an infinite sequence is either n or a smaller integer. Thus, in the example, (1111) has primitive period 1.

The author uses $F_q(n)$ to denote the number of equivalence classes with primitive period n and $F_q^*(n)$ to denote the total number of classes with period n (whether primitive or not). He then shows how to compute $F_q(n)$, but the formulae are too complicated to quote here.

R. G. Stanton (Waterloo, Ont.)

3795:

Freudenthal, Hans. Ein kombinatorisches Problem von biochemischer Herkunft. Nederl. Akad. Wetensch. Proc. Ser. A 61 = Indag. Math. 20 (1958), 253–258.

For n and k fixed, consider the n^k words $(a_1 a_2 \cdots a_k)$ formed from the "alphabet" $1, 2, \cdots, n$. Golomb, Gordon, and Welch [Canad. J. Math. 10 (1958), 202–209; MR 20 #1597] define a comma-free dictionary as a set D of k-letter words such that whenever $(a_1 a_2 \cdots a_k)$ and $(b_1 b_2 \cdots b_k)$ are in D, then the "overlaps" $(a_2 a_3 \cdots a_k b_1), \cdots, (a_k b_1 \cdots b_{k-1})$ are not in D. They further denote the maximum number of words in D by $W_k(n)$; it is immediate that $W_3(4)=20$.

The present paper shows, by enumeration, that there are 5 comma-free dictionaries containing 20 words in this one case $n=4$, $k=3$. These are the example given by Golomb, Gordon and Welch [ibid.], comprising all words (abc) with $a<b\geqq c$, and four minor variations of this example, which differ from it in one, two, three, and four words, respectively.

{No weight should be attached to the attempt at a biochemical interpretation based on the claim that there are only 20 natural amino-acids.}

R. G. Stanton (Waterloo, Ont.)

3796:

Lotkin, Mark. The partial summation of series by matrix methods. Amer. Math. Monthly 64 (1957), 643–647.

Four identities are considered: $\sum_{i=1}^n (-1)^{n-i}\binom{n+i-1}{n}\cdot\binom{n}{i}=1$, $\sum_{i=1}^n(-1)^i(i+j-1)^{-1}\binom{n+i-1}{n}\binom{n}{i}=0$, $2\leqq j\leqq n$, and two more complicated ones of the form $A_{njk}\sum_{i=1}^n c_{nijk}=\delta_{jk}$. It is shown how certain well-known formulas in finite differences and interpolation lead to these identities; e.g., $\sum_{i=0}^r (-1)^i f(i)\binom{r}{i}=\delta_{nr}(-1)^n n! a_n$, where $f(x)=\sum_0^n a_k x^k$. The identities themselves are deduced by matrix methods: let $c_{nijk}=a_{nij}\cdot\alpha_{nik}$ and use some of the author's matrix inversion formulas [Math. Tables Aids Comput. 9 (1955), 153–161; MR 17, 667]. Applicability of the method is discussed and alternative proofs of the identities given. *Z. A. Melzak* (Montreal, P. Q.)

ORDER, LATTICES

See also 3773, 3959.

3797:

Leenders, J. H. Birkhoff's problem 103. Simon Stevin 32 (1958), 1–22.

Brute force has its day in this exhaustive discussion of the various l-ring structures which can be imposed on the lexicographically ordered two-dimensional real vector space. A less complete (but more succinct) discussion of these same l-rings is given in Example 9, p. 48, of Birkhoff and Pierce's paper in An. Acad. Brasil. Ci. 28 (1956), 41–69 [MR 18, 191]. *R. S. Pierce* (Seattle, Wash.)

3798:

Kurepa, Đuro. Partitive sets and ordered chains. Rad Jugoslav. Akad. Znan. Umjet. Odjel. Mat. Fiz. Tehn. Nauke 6 (302) (1957), 197–235. (Serbo-Croatian summary)

This paper contains results about partial and total orderings, similarity and "strong" similarity, maximal chains, choice axiom, and related matters. Among the more easily stated results are the following. Let PM denote the set of all subsets of M, ordered by inclusion; $I(\alpha)$ the set of all ordinals $<\alpha$; $\nu(\alpha)$ the lexicographically ordered set of all mappings of $I(\alpha)$ into $I(\nu)$. Theorem 2.1: A chain is similar to a subset of PM if and only if it has a dense set of power card M and at most card M one-sided limit points. Theorem 4.1: The natural one-one mapping from $PI(\alpha)$ onto $2(\alpha)$ is order-preserving, i.e., the order on $PI(\alpha)$ can be extended to a total order by defining $X<Y$ to mean that the first element in the symmetric difference belongs to Y. Theorem 5.2: For fixed α, any two sets $n(\omega_\alpha)$, for finite $n\geqq 2$, are similar; and any two sets $\nu(\omega_\alpha)$, for $2\leqq\nu\leqq\omega_\alpha$, contain copies of one another.

L. Gillman (Princeton, N.J.)

3799a:

Petropavlovskaya, R. V. Associative systems that are lattice-isomorphic to a given group. II. Vestnik Leningrad. Univ. 11 (1956), no. 19, 80–99. (Russian)

3799b:

Petropavlovskaya, R. V. Semigroups lattice-isomorphic to groups. III. Vestnik Leningrad. Univ. 12 (1957), no. 19, 5–19. (Russian. English summary)

Continuation of an earlier paper [same Vestnik 11 (1956), no. 13, 5–26; MR 18, 555]. Theorem: In order that a group G be lattice isomorphic to a semigroup which is not a group it is necessary and sufficient that: (i) G has only elements of finite order; (ii) G has a central cyclic Sylow subgroup S such that the subgroup generated by all Sylow subgroups $\neq S$ is a proper subgroup of G.

Finally, conditions that a semigroup be lattice-isomorphic to a group are established.

W. T. van Est (Utrecht)

3800:

Goffman, Casper. Remarks on lattice ordered groups and vector lattices. I. Carathéodory functions. Trans. Amer. Math. Soc. **88** (1958), 107–120.

Le principal résultat démontré par l'auteur est le suivant: Tout groupe réticulé G conditionnellement complet (non supposé abélien) peut être réalisé comme sous-groupe ordonné coréticulé d'un espace vectoriel (réel) réticulé.

Ceci entraîne que tout groupe réticulé archimédien admet une telle réalisation et, en particulier, est abélien. L'auteur obtient un tel résultat en associant à tout treillis distributif relativement complémenté L un espace vectoriel réticulé $C(L)$ (dit espace des fonctions de Carathéodory engendrées par L). Il réalise G comme sous-groupe de $C(G^*)$, G^* étant le treillis des filets de G.

Une semi-norme p sur un espace vectoriel réticulé X est dite compatible si $|x|\geqq|y|$ entraine $p(x)\geqq p(y)$ ($|x|=(x\cup 0)-(x\cap 0)$ si $x\in X$). L'auteur termine en discutant l'existence de semi-normes compatibles sur certains espaces vectoriels réticulés de type $C(L)$. Il montre en particulier qu'il n'y a pas de semi-norme compatible non triviale sur l'espace X des classes de fonctions mesurables sur l'intervalle $[0, 1]$.

P. Jaffard (Paris)

3801:

Goffman, Casper. A class of lattice ordered algebras. Bull. Amer. Math. Soc. **64** (1958), 170–173.

In a recent paper by the same author [#3800 above], a representation theorem for lattice ordered groups by Carathéodory functions was given. The present paper studies the analogous problem for lattice ordered algebras. The author's definition of Carathéodory functions generated by a relatively complemented distributive lattice is a cross between the original one of Carathéodory [Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1938**, no. 1, 27–68] and the one given by D. A. Kappos [Math. Z. **51** (1949), 616–634; MR **10**, 437]. For its complete description, see the author's paper reviewed above. In the definition of Carathéodory functions as well as in the statement of the main result, an essential role is played by the notion of carrier, introduced by P. Jaffard [J. Math. Pures Appl. (9) **32** (1953), 203–280; MR **15**, 284] under the name of filet. For two positive elements x, y (i.e. $x\geqq 0$, $y\geqq 0$) in a lattice-ordered group, we say $x\sim y$, if $x\cap z=0$ when and only when $y\cap z=0$. The equivalence classes thus obtained for positive elements are called carriers and form a lattice. The carrier of a positive element x is the equivalence class containing x. The main result is the following theorem. A lattice-ordered algebra A is isomorphic with the algebra of Carathéodory functions generated by a relatively complemented distributive lattice, if and only if A satisfies these three conditions: (1) A is conditionally complete; (2) every sequence of pairwise disjoint positive elements of A whose carriers have an upper bound, itself has an upper bound; (3) for positive elements x, y, z of A, the relation $(xy)\cap z=0$ is equivalent to $x\cap y\cap z=0$.

Ky Fan (Notre Dame, Ind.)

3802:

Nikodým, Otton Martin. Critical remarks on some basic notions in Boolean lattices. II. Rend. Sem. Mat. Univ. Padova **27** (1957), 193–217.

[For part I, see An. Acad. Brasil. Ci. **24** (1952), 113–136; MR **14**, 126]. The author discusses some basic notions which he believes are not treated adequately in the previous literature.

I. Halperin (Kingston, Ont.)

3803:

Wright, Fred B. Some remarks on Boolean duality. Portugal. Math. **16** (1957), 109–117.

A consequence of a theorem by Stone [Trans. Amer. Math. Soc. **40** (1936), 37–111] is that a homomorphism from one Boolean algebra into another is induced by a continuous function from a suitable Boolean space into another. Halmos [Compositio Math. **12** (1956), 217–249; MR **17**, 1172] extended this correspondence to hemimorphisms, (which satisfy weaker conditions than homomorphisms); viz., every hemimorphism is induced by a Boolean relation. In this paper the author defines, for any mapping whatsoever between Boolean algebras, a dual relation between the corresponding Boolean spaces.

B. A. Galler (Ann Arbor, Mich.)

GENERAL ALGEBRAIC SYSTEMS

See also 3786.

3804:

★Bourbaki, N. Éléments de mathématique. 22. Première partie: Les structures fondamentales de l'analyse. Livre 1: Théorie des ensembles. Chapitre 4: Structures. Actualités Sci. Ind. no. 1258. Hermann, Paris, 1957. 125 pp.

This chapter is devoted to the introduction of some basic algebraic concepts, such as isomorphism, homomorphism, free system, free product, direct product, direct limits, and inverse limit. The systems considered consist of finitely many principal base sets and finitely many auxiliary base sets, together with relations of finite rank over these sets, or possibly relations of a higher type. Due to the extreme generality, the definitions are rather cumbersome, and all the results derived are of a very trivial nature. Approximately half the chapter is devoted to historical notes pertaining to Chapters 1–4.

B. Jónsson (Minneapolis, Minn.)

3805:

Mal'cev, A. I. Defining relations in categories. Dokl. Akad. Nauk SSSR **119** (1958), 1095–1098. (Russian)

The paper presents an approach to definition by generators and relations in arbitrary (concrete) categories of sets and functions. The newly defined terms are, accordingly, numerous. The main stress (Theorems 3–5) is on the concept of a "replica" $\mathfrak{A}^L$ of an object $\mathfrak{A}$ of the category K, in the subcategory L. This is an object in L with a mapping $\mathfrak{A}\rightarrow\mathfrak{A}^L$ such that every mapping $\mathfrak{A}\rightarrow\mathfrak{B}$, $\mathfrak{B}$ in L, factors $\mathfrak{A}\rightarrow\mathfrak{A}^L\rightarrow\mathfrak{B}$. A typical result is that if $\mathfrak{A}$ is the free product in L of the objects $\mathfrak{A}_\alpha$ of K, and each $\mathfrak{A}_\alpha$ has a replica $\mathfrak{A}_\alpha{}^L$, then $\mathfrak{A}$ is the free product in L of the objects $\mathfrak{A}_\alpha{}^L$.

J. Isbell (Seattle, Wash.)

THEORY OF NUMBERS

3806:

Cellitti, Carlo. Nuove dimostrazioni del teorema di Fermat, del teorema di Fermat generalizzato da Eulero, e del teorema di Wilson. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 121–123.

The proof of Euler's generalization of Fermat's theorem

is essentially that given by Uspensky and Heaslet [Elementary number theory, McGraw-Hill, New York, 1939; MR 1, 38; p. 150]; the proof of Wilson's theorem, by means of primitive roots, is given in the same book, p. 231.
I. A. Barnett (Dublin)

3807:

Trustrum, G. B. On sequences of integers. Mathematika 5 (1958), 38–39.

The author constructs two infinite sequences of integers $\{a_i\}$, $\{b_j\}$, $1\leq i<\infty$, $1\leq j<\infty$ so that the sequence $\{a_i+b_j\}$, $1\leq i$, $j<\infty$, contains no infinite subsequence of which no term divides any other. This disproves a conjecture of the reviewer.
P. Erdős (Haifa)

3808:

Hazanov, M. B. Formulas for Pythagorean numbers. Kabardin. Gos. Ped. Inst. Uč. Zap. 12 (1957), 14. (Russian)

An elementary method is given for determining all Pythagorean triples x, y, z; that is, such that $x^2+y^2=z^2$.

3809:

Eljoseph, Nathan. On polynomials integral for integral values of the argument. Riveon Lematematika 11 (1957), 41–46. (Hebrew. English summary)

Defining $\pi_0(x)=1$, $\pi_n(x)=\dfrac{x(x-1)\cdots(x-n+1)}{n!}=\dbinom{x}{n}$ we see that for every $n\geq 2$ there exist polynomials having integral values for integral values of the argument even though not every coefficient is integral.

It is proved that a polynomial has integral values for integral values of the argument if and only if when expressed in the form $\sum a_i\pi_i(x)$ (which is always possible) all a_i are integral. It follows that when a polynomial of degree n is integral for $n+1$ consecutive integral values of the argument it is integral for all integral values of the argument. Five corollaries are given.
From the author's summary

3810:

Streefkerk, H. A special trigonometric sum and the Möbius function. Nieuw Tijdschr. Wisk. 45 (1957/58), 275–279. (Dutch)

Proof that the sum $\sum_j{}' \cos(2\pi/jn)$, where j runs through a reduced system modulo n, is equal to $\mu(n)$. This result is of course well-known.
J. Popken (Amsterdam)

3811:

van der Corput, J. G. Inequalities involving least common multiple and other arithmetical functions. Nederl. Akad. Wetensch. Proc. Ser. A 61=Indag. Math. 20 (1958), 5–15.

Let $P_1, \cdots, P_n$ be distinct primes, $A=P_1\cdots P_n$, $\{u_1, \cdots, u_n\}$ the least common multiple of the positive integers $u_1, \cdots, u_n$, $\Lambda(u_1, \cdots, u_n)=\{u_1, \cdots, u_n\}^{-r}$ $(r\geq 0)$. For each (positive) divisor B of A introduce a real number $\lambda(B)$ and put $\sigma(B)=\sum_{D|B}\lambda(D)$ and $\tau(B)=\sum_{D|B}\lambda(D)\Lambda(a_1^{\delta_1}, \cdots, a_n^{\delta_n})$, where $D=P_1^{\delta_1}\cdots P_n^{\delta_n}$ and $a_1, \cdots, a_n$ are positive integers. The main theorem is in eight parts, too lengthy to be given in full, but the following two are typical. (2) If $\sigma(D)\leq\sigma(B)$ for each divisor B of A and each divisor D of B, then $\tau(D)\leq\tau(B)$ for each divisor B of A, for each divisor D of B and for each choice of the positive integers $a_1, \cdots, a_n$. (3) If $\sigma(B)+\sigma(C)\leq\sigma(BC)$ for any two positive integers B and C whose product divides A, then $\tau(B)+\tau(C)\leq\tau(BC)$ for any B, C whose product divides A and for each choice of $a_1, \cdots, a_n$. A typical consequence is the inequality

$$\sum_{D|B}\mu(D)\{a_1^{\delta_1}, \cdots, a_n^{\delta_n}\}^{-r}\cdot\sum_{D|C}\mu(D)\{a_1^{\delta_1}, \cdots, a_n^{\delta_n}\}^{-r} \leq\sum_{D|BC}\mu(D)\{a_1^{\delta_1}, \cdots, a_n^{\delta_n}\}^{-r},$$

where B, C are relatively prime quadratfrei numbers and μ is the Möbius function. For $r=1$ this reduces to an inequality of the reviewer [Bull. Amer. Math. Soc. 54 (1948), 681–684; MR 10, 104]. Further results are obtained for more general functions Λ and for the case where A is no longer quadratfrei.
F. A. Behrend (Victoria)

3812:

Carlitz, L. Some cyclotomic determinants. Bull. Calcutta Math. Soc. 49 (1957), 49–51.

If p is a prime ≥ 3 and $k|(p-1)$ $(k>1)$, if g is a fixed primitive root (mod p) and α is a fixed kth root of unity, the symbol $\left(\dfrac{r}{p}\right)_k$ is defined as being $\alpha^{\operatorname{ind} r}$ if p does not divide r, and zero otherwise, where ind r is the index to base g. The author shows that

$$D_k=\left|\left(\frac{r-s}{p}\right)_k\right| \quad (r, s=0, 1, 2, \cdots, p-2)$$

can be evaluated, in terms of the cyclotomic functions

$$\psi_r(\alpha)=\sum_{s=1}^{p-2}\alpha^{\operatorname{ind} s-(r+1)\operatorname{ind}(s+1)}$$

as $D_k=p^{t-1}[\psi_1(\alpha)\cdots\psi_{k-2}(\alpha)]^t$, where $kt=p-1$. Further, if $\varepsilon=e^{2\pi i/p}$, $\eta_r=\sum_{s=0}^{t-1}\varepsilon^{g^{r+sk}}$, th edeterminant $\Delta_k=|\eta_{r-s}|$ $(r, s=0, 1, \cdots, k-1)$ may be evaluated as $\Delta_k=-p^{(k-1)/2}$ (k odd) or $\Delta_k=(-1)^{(p-1)(p+k-3)/4}p^{(k-1)/2}$ (k even).
R. G. Stanton (Waterloo, Ont.)

3813:

Bhaskaran, M. On some theorems on congruence. Math. Student 25 (1957), 40–41.

The author gives a new proof of the following theorem of P. Kesava Menon [J. Indian Math. Soc. 2 (1937), 332–333]: If p is an odd prime, $1^{p-1}+2^{p-1}+\cdots+(p-1)^{p-1}\equiv p+(p-1)!$ (mod p^2). He also proves some other results relating to symmetric functions modulo p, which are contained in theorems of M. Bauer [Mat. Phys. Lapok 12 (1903), 159–160] and U. Concina [Period. Mat. 28 (1913), 164–177, 267–270].
R. J. Levit (San Francisco, Calif.)

3814:

Hoehnke, Hans-Jürgen. Identische Kongruenzen für Polynome nach zusammengesetzten Moduln. Math. Nachr. 15 (1956), 141–154.

For powers of an odd prime p the author proves identities which include generalizations by M. Bauer and S. Lubelski of the Lagrange identity

$$(y-1)(y-2)\cdots(y-p+1)\equiv y^{p-1}-1 \pmod p,$$

where y is an indeterminate. His most general result in this connection involves an arbitrary subgroup U of the group P_μ of residues modulo p^μ which are prime to p, an arbitrary positive integer n, and an arbitrary polynomial $f(x)$ whose coefficients are themselves polynomials with rational integral coefficients in a finite number of indeterminates independent of x. Let g be the order of U. If d denotes the g.c.d. of g and n, then g/d is the order of the group of nth powers of the elements of U, and $g/d=fp^\nu$, where f is a factor of $p-1$ and $\nu\leq\mu-1$. Define $C_0, \cdots, C_{f-1}$ by the identity $f(x)=(x^{f-1}-1)/_1(x)+C_0+C_1x+\cdots+C_{f-1}x^{f-1}$, and let Z

denote the cyclic determinant

$$Z=\begin{vmatrix} y+C_0, & C_1, & \cdots, & C_{f-1} \\ C_{f-1}, & y+C_0, & \cdots, & C_{f-2} \\ \cdots & \cdots & \cdots & \cdots \\ C_1, & C_2, & \cdots, & y+C_0 \end{vmatrix}.$$

The author's Theorem 12 states that

$$\prod_{u\in U} [y+f(u^n)] \equiv Z^{\varphi/f} \pmod{p^\mu}.$$

This reduces to Lagrange's identity if $\mu=1$, $n=1$, $U=P_\mu$, $f(x)=-x$; it reduces to the Bauer generalization, for an odd prime power modulus, if $\mu\geqq 1$, $n=1$, $U=P_\mu$; it reduces to a result of Lubelski if $U=P_\mu$, $y=0$, and the coefficients of $f(x)$ are rational integers. The author also proves corresponding identities for powers of 2 and discusses the obtaining of identities for a general modulus from those for its prime power factors.
R. Hull (Pacific Palisades, Calif.)

3815:

Cestari, R. Risoluzione della diofantea $X^y-Z^t=1$. Giorn. Mat. Battaglini (5) 5 (85) (1957), 197–208.

This paper purports to prove "Catalan's conjecture" that the equation in the title has no integer solutions other than $3^2-2^3=1$. There is a fallacy in the remark B_1 below equation (20) on page 207.
J. W. S. Cassels (Cambridge, England)

3816:

Nilov, G. N. The number of non-negative integer solutions of the equation $\sum_{k=1}^{n-1} x_k+p\cdot x_n=m$. Kabardin. Gos. Ped. Inst. Uč. Zap. 12 (1957), 21–23. (Russian)

3817:

Tkačev, M. I. The number of non-negative integer solutions of the equation $\sum_{i=1}^{l} i\cdot x_i=m$. Kabardin. Gos. Ped. Inst. Uč. Zap. 12 (1957), 75–82. (Russian)

3818:

Tkačev, M. I. Calculation of certain products. Kabardin. Gos. Ped. Inst. Uč. Zap. 12 (1957), 83–85. (Russian)

3819:

Tkačev, M. I. The generating function $p_n(m)$. (Method and formulas for $n=8, 9$). Kabardin. Gos. Ped. Inst. Uč. Zap. 12 (1957), 86–98. (Russian)

The function $p_n(m)$ is defined as the number of solutions in natural numbers x_i of the Diophantine equation $x_1+2x_2+3x_3+\cdots+nx_n=m$.

An inductive method is presented for calculating $p_{n+1}(m)$ from $p_n(m)$, and finite, closed (rather lengthy) formulas are obtained, for the first time, for $n=8, 9$.

3820:

Labutin, D. N. Refinement of an algorithm of I. Čistyakov. Kabardin. Gos. Ped. Inst. Uč. Žap. 12 (1957), 63–64. (Russian)

The article points out that without proper care some of the solutions of the Diophantine equation

$$ax^2+bxy+cy^2+dx+ey=0$$

with integral non-zero coefficients may be lost in making the substitution $y=xt$.

3821:

Palamà, Giuseppe. Su di una congettura di Sierpiński relativa alla possibilità in numeri naturali della $5/n=1/x_1+1/x_2+1/x_3$. Boll. Un. Mat. Ital. (3) 13 (1958), 65–72.

W. Sierpiński [Mathesis 65 (1956), 16–32; MR 17, 1185] conjectured that the equation

$$\frac{5}{n}=\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}$$

is soluble in positive integers x_1, x_2, x_3, for every positive integer n, and verified that this is so for $n\leqq 1000$. By a detailed consideration of the form of n, and with the aid of a criterion due to G. Mignosi [Rend. Sem. Fac. Sci. Univ. Cagliari 1 (1931), 97–107], the author proves that this equation is always soluble if n is not of the form $1260m+1$, and verifies that, even for n of this form, it is soluble if $n\leqq 922321$.
R. A. Rankin (Glasgow)

3822:

Cellitti, Carlo. Sopra una costruzione di sistemi di rappresentanti di classi relative a G_2 di forme quadratiche binarie primitive di seconda specie di determinante $D>0$ e $\equiv 5$ (mod. 8). Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 23 (1957), 15–21.

The author continues his painstaking rediscovery of one of the less ancient ways of relating the number of properly and improperly quadratic forms of given determinant [cf. Boll. Un. Mat. Ital. (3) 10 (1955), 527–530; same Atti (8) 21 (1956), 57–60; MR 17, 827; 18, 641].
J. W. S. Cassels (Cambridge, England)

3823:

Venkatachalam Iyer, R. Sur les formes concordantes. Mathesis 66 (1957), 138–144.

Two forms ax^2+by^2, cx^2+dy^2 are called concordant (Euler) if they can be made simultaneously square for certain positive integers x and y. Various known cases of concordant forms are given, and a number of new cases are found. For example, it is shown that if $a+b$ and $c+d$ are squares, then the forms are concordant. In fact, the forms are squares for $x=4bd(a+b)(c+d)-(ac-bd)^2$ and $y=4ac(a+b)(c+d)-(ac-bd)^2$. As yet unsolved is the problem of finding all n for which x^2+y^2 is concordant with x^2+ny^2.
L. Moser (Edmonton, Alta.)

3824:

Birch, B. J.; and Davenport, H. Quadratic equations in several variables. Proc. Cambridge Philos. Soc. 54 (1958), 135–138.

The following theorem, which has applications in the work of the authors on the representation of arbitrarily small numbers by quadratic forms in many variables, is proved. Theorem: Let $f(x_1, \cdots, x_n)$ be an indefinite quadratic form with integral coefficients, and let $g(x_1, \cdots, x_n)$ be a positive definite form with real coefficients. If the equation $f=0$ is properly soluble in integers, then it has a solution satisfying

$$0<g(x_1, \cdots, x_n)\leqq(\gamma_{n-1})^{n-1}(2\,\mathrm{Tr}(\mathbf{fg}^{-1})^2)^{\frac{1}{2}(n-1)}(\det g).$$

Here $\mathbf{f}$ resp. $\mathbf{g}$ denotes the matrix of coefficients of the form f resp. g, Tr is the trace of a matrix, and γ_{n-1} is Hermite's constant which occurs in the theory of the minima of definite quadratic forms.

The proof is a modification of one given by Davenport [same Proc. 53 (1957), 539–540; MR 19, 125] of a result by the reviewer [ibid. 51 (1955), 262–264; 52 (1956), 602; MR 16, 1002; 18, 380], of which the theorem here is a generalization.
J. W. S. Cassels (Cambridge, England)

3825:

Ennola, Veikko. On the first inhomogeneous minimum of indefinite binary quadratic forms and Euclid's algorithm in real quadratic fields. Ann. Univ. Turku. Ser. AI **28** (1958), 58 pp.

Let $f(x, y)=ax^2+bxy+y^2$ be an indefinite quadratic form with discriminant $d=b^2-4ac$. In his chapter I the author shows that there exist real numbers x_0, y_0 such that

$$|f(x_1, y_1)|\geqq\kappa d^{\frac{1}{2}}$$

for all numbers $x_1\equiv x_0 \pmod 1$, $y_1\equiv y_0 \pmod 1$, where $\kappa=(16+6\sqrt{6})^{-1}\doteqdot(30.69)^{-1}$. This value of κ is a very considerable improvement on the best hitherto known [Cassels, Proc. Cambridge Philos. Soc. **48** (1952), 72–86, 519–520; MR **13**, 919]. The proof is a modification of the original method of Davenport [Proc. London Math. Soc. (2) **53** (1951), 65–82; MR **13**, 15], which involves developments in continued fractions to the nearest integer(!). It is remarkable that the author can give a simple prescription for finding x_0, y_0 in terms of Davenport's algorithm.

In the rest of his paper the author gives a systematic determination of all the real quadratic fields with a euclidean algorithm. In chapter II he shows that all the fields with a euclidean algorithm do in fact possess it, and in chapter III he shows that no other fields have it. It is very useful to have such a systematic treatment of results otherwise scattered through the literature and there attacked by diverse methods.

J. W. S. Cassels (Cambridge, England)

3826:

***Walfisz, Arnold. Gitterpunkte in mehrdimensionalen Kugeln.** Monografie Matematyczne. Vol. 33. Państwowe Wydawnictwo Naukowe, Warsaw, 1957. 471 pp.

This is a beautifully written book by a leading expert in the field. Although of immense value to the specialist, it is addressed to a wider circle of readers. To quote the author's own words, "Das Studium des Buches setzt nur Kenntnisse voraus, wie sie in den üblichen Anfängervorlesungen über Analysis, Algebra und Zahlentheorie an den Hochschulen gegeben werden. Auch sind die Rechnungen überall sehr eingehend durchgeführt." Almost a third of the book is devoted to researches of the last ten years.

The book is concerned with the study of $P_k(X)$, which is the difference between the number of lattice-points in the k-dimensional hyper-sphere (1) $y_1^2+y_2^2+\cdots+y_k^2\leqq X$ and its volume $V_k(X)$. So (2) $P_k(X)=A_k(X)-V_k(X)$, where $A_k(X)$ is the number of lattice-points satisfying (1). It is well known that we have the asymptotic relation $A_k(X)\sim V_k(X)$.

Gauss observed that $P_2(X)=O(X^{\frac{1}{2}})$. Sierpiński (1909) found $P_2(X)=O(X^{\frac{1}{3}})$, van der Corput proved the sharper result $P_2(X)=O(X^c)$ with $c<1/3$, and there have been petty improvements in the exponent in later years. It is also known (Hardy) that the exponent cannot be lowered below 1/4; on the other hand it is considered highly probable that an exponent $1/4+\varepsilon$, where $\varepsilon>0$ is arbitrary, represents the truth. We proceed to sketch the contents of the book chapter by chapter.

Chapter 1 contains Estermann's [J. London Math. Soc. **20** (1945), 66–67; MR **7**, 414] well-known short-cut to the determination of the sign in the Gaussian sum. Next, the author obtains Jacobi's formula for the number of representations of a number as a sum of 4 squares. The treatment, involving infinite series, was rediscovered by Ramanujan [Trans. Cambridge Philos. Soc. **22** (1916), 159–184]. The chapter concludes with a formula of Landau for $A_k(X)$ and the well-known "singular series" representation of Hardy for $r_k(n)$, the number of representations of n as a sum of k squares, when $k>4$.

Ch. 2 contains elementary estimates of $P_k(X)$. H. Weyl's method of estimating exponential sums is used to obtain $P_4(X)=O(X \log X/\log\log X)$, which is due to the author [Math. Z. **26** (1927), 66–88]. This chapter concludes with the deep result obtained from L. K. Hua's and Vinogradoff's methods: $P_4(X)=O(X \log^{\frac{2}{3}} X(\log\log X)^{\frac{1}{2}})$.

Ch. 3 contains "Ω-results" (the Ω-notation is due to Littlewood), e.g.

$$P_k(X)=\Omega(X^{k/2-1});\quad P_4(X)=\Omega(X \log\log X).$$

The rest of the chapter studies the function $\rho_k(t)=P_k(t)/t^{k/2-1}$ ($\rho_k(X)$ is bounded for $k>4$ as $X\to\infty$). In particular, we have the result that the sequence of numbers $\rho_k(1)$, $\rho_k(2)$, $\rho_k(3)$, $\cdots$ has infinitely many limit points for $k\geqq5$.

Chs. 4 and 5 contain results of Petersson [Abh. Math. Sem. Hamburg. Univ. **5** (1926), 116–150] and Lursmanaschwili-Walfisz. These are too complicated to be quoted here. Chs. 6 and 7 have more on the functions P_k and ρ_k for even k and odd k, respectively. Ch. 8: $\int_0^X P_4^2(y)dy=\frac{2}{3}\pi^2X^3+O(X^{5/2})$. This is heavy going — a formidable exercise in analytic number theory! The result is due to Walfisz. Ch. 9: $\int_0^X P_k^2(y)dy$ (Jarnik). Ch. 10: Development of $P_k(t)$ in Bessel function series.

The author remarks that the study of ellipsoids $\alpha_1X_1^2+\alpha_2X_2^2+\cdots+\alpha_kX_k^2\leqq Y$ with irrational coefficients would require a separate monograph. {The reviewer would also like to mention in this connection the striking recent work of Davenport, Heilbronn, and G. L. Watson on irrational indefinite quadratic forms and that of Birch and D. J. Lewis [#3828, #3827 below] on the nontrivial representation of 0 by "mixed" cubic forms (with coefficients in any algebraic number field) with a sufficient number of variables.} *S. Chowla* (Princeton, N.J.)

3827:

Lewis, D. J. Cubic forms over algebraic number fields. Mathematika **4** (1957), 97–101.

The author proves: For every algebraic number field K and for all integers $h>0$, $m\geqq0$, there exists an integer $\Psi(K, h, m)$ such that every system of h cubic homogeneous polynomial equations in more than $\Psi(K, h, m)$ variables has an m-dimensional linear manifold of solutions in K. The proof depends on R. Brauer's reduction to the case of diagonal forms ($\sum \alpha_j y_j^3$) [Bull. Amer. Math. Soc. **51** (1945), 749–755; MR **7**, 108], applied over the field $K((-1)^{\frac{1}{2}})$ instead of K; use of the fact that every quadratic form in 5 variables over $K((-1)^{\frac{1}{2}})$ has a nontrivial zero; a result of L. G. Peck on diagonal forms [Amer. J. Math. **71** (1949), 387–402; MR **10**, 515]; and a lemma showing that if a cubic form over K has a nontrivial zero in $K((-1)^{\frac{1}{2}})$ then it has a nontrivial zero in K. A further lemma shows that the result of Peck is needed only for the field $Q((-1)^{\frac{1}{2}})$ (where Q=rational field), rather than for arbitrary fields K.

G. Whaples (Bloomington, Ind.)

3828:

Birch, B. J. Homogeneous forms of odd degree in a large number of variables. Mathematika **4** (1957), 102–105.

The author proves the following. Let $h\geqq1$ and $m\geqq1$ be integers and let $r_1, \cdots, r_h$ be odd positive integers. Let K be an algebraic number field. Then there exists a number

$\Psi(r_1, \cdots, r_h; m; K)$ such that every system of h homogeneous forms of degrees $r_1, \cdots, r_h$, respectively, in more than $\Psi(r_1, \cdots, r_h; m; K)$ variables, has an m-dimensional linear space of simultaneous solutions in K. The methods are very similar to those of Lewis, described in the preceding review, together with an induction on the maximum degree. *G. Whaples* (Bloomington, Ind.)

3829:

Peyser, Gideon. On the domain of absolute convergence of Dirichlet series in several variables. Proc. Amer. Math. Soc. **9** (1958), 545–550.

Verf. untersucht die Bereiche der absoluten und gewöhnlichen Konvergenz von Dirichletreihen z_k der Form (1) $\sum_{m=1}^{\infty} A_m \exp(-\sum_{k=1}^{n} \sigma_{mk} z_k)$, mit n komplexen Veränderlichen und komplexen A_m und σ_{mk}. In Ausdehnung des bekannten Falles $n=1$ [J. F. Ritt, Trans. Amer. Math. Soc. **18** (1917), 27–49; E. Hille, Ann. of Math. (2) **25** (1924), 261–278] wird gezeigt, daß der Bereich der absoluten Konvergenz der Reihe (1) konvex ist. Ein weiteres Ergebnis über den Zusammenhang zwischen absoluter und gewöhnlicher Konvergenz verallgemeinert und verschärft im Falle $n=1$ ein Ergebnis von T. M. Gallie, Jr. [dieselben Proc. **7** (1956), 627–629; MR **18**, 124] und enthält den klassischen Satz für $n=1$ mit reellen Koeffizienten. *H.-E. Richert* (Göttingen)

3830:

Erdös, Paul; and Shapiro, Harold N. On the least primitive root of a prime. Pacific J. Math. **7** (1957), 861–865.

Let $g(p)$ be the positive least primitive root mod p. The authors prove that $g(p)=O(m^c p^{\frac{1}{2}})$, where c is a constant and m is the number of distinct prime factors of $p-1$. They use Brun's method and the following lemma concerning character sums. Let $\chi(n)$ be a non-principal character, mod p; then $|\sum \chi(u+v)| \leq (pN(U)N(V))^{\frac{1}{2}}$, where u and v run over the sets U and V of distinct integers mod p, and $N(U)$, $N(V)$ are the number of integers in the sets. *L. K. Hua* (Peking)

3831:

Heilbronn, H. On the averages of some arithmetical functions of two variables. Mathematika **5** (1958), 1–7.

It is shown that

$$\sum_{p\leq x} \sum_{q\leq x} \left(\frac{p}{q}\right) = O(x^{7/4}(\log x)^{-3/4}),$$

where p and q are restricted to odd primes and $\left(\frac{p}{q}\right)$ denotes the Legendre symbol. A similar estimate is obtained for the more general sum, $\sum_{n=1}^{N} \sum_{p\leq x} \chi_n(p)$, where $\chi_1, \cdots, \chi_N$ are distinct Dirichlet characters. This latter result is generalized to algebraic number fields and then used to prove the following theorem: If k and l are two natural numbers, then the number of pairs of primes p and q not exceeding x which satisfy $p\equiv 1 \pmod k$, $q\equiv 1 \pmod l$ and for which the congruences $y^k \equiv q \pmod p$, $z^l \equiv p \pmod q$ are solvable in rational integers y and z, equals

$$(kl\varphi(k)\varphi(l))^{-1} v \left\{\int_e^x (\log u)^{-1} du\right\}^2 + O(x^2)\exp(-\alpha(\log x)^{\frac{1}{2}}),$$

where $v=1$ if $(2, k, l)=1$, $v=3/2$ if $k\equiv l\equiv 2 \pmod 4$, $v=2$ otherwise, and α and the constant implied by the symbol O depend on k and l only.
W. H. Mills (New Haven, Conn.)

3832:

Christopher, John. The asymptotic density of some k-dimensional sets. Amer. Math. Monthly **63** (1956), 399–401.

Let A be a subset of the set of k-tuples $(a_1, \cdots, a_k)$ of positive integers, where $k \geq 1$. Let A_n be the number of elements in A with every $a_i \leq n$. Then the asymptotic density $\delta(A)$ of A (namely, the probability that a random k-tuple of positive integers will be in A) is defined by $\delta(A)=\lim_{n\to\infty} \{A_n/n^k\}$.

The author proves that if A is the set of k-tuples whose greatest common divisor is square-free, then $\delta(A)= 1/\zeta(2k)$, where ζ is the Riemann zeta-function, and if, for $k>1$, A is the set of k-tuples whose g.c.d. is t, with $t\geq 1$, then $\delta(A)=t^{-k}/\zeta(k)$. *S. H. Gould* (Providence, R.I.)

3833:

Muchart, P. Sur un théorème de Fermat. Mathesis **67** (1958), 131–134.

"Fermat, en énonçant son célèbre théorème: tout nombre premier $4n+1$ est somme de deux carrés, a seulement indiqué la méthode générale suivie (preuve à la descente), mais non la démonstration. Si celle-ci n'a pas été fait par la suite, peut-on en imaginer une basée sur les connaissances probables de Fermat? Nous essaierons de répondre à cette question, en indiquant en langage moderne et en nous appuyant sur des propriétés élémentaires qu'il devait, pensons-nous, connaître, sinon comme il a procédé, du moins comment il aurait pu le faire."
Résumé de l'auteur

3834:

Lavrik, A. F. Addition of a prime to a prime power of a given prime. Dokl. Akad. Nauk SSSR **119** (1958), 1085–1087. (Russian)

The article discusses questions of the sort: for how many n, not greater than a prescribed bound, is the equation

$$n=p+P^q$$

solvable in prime numbers p, P, q, where P is fixed.

3835:

Vinogradov, A. I. On the connections between the sieve of Eratosthenes and the Riemann ζ-function. Vestnik Leningrad. Univ. **11** (1956), no. 13, 142–146. (Russian)

The author sketches a proof, the full details of which are being sent to the Mat. Sbornik [see #3836 below], of the result that every sufficiently large even integer N can be written in the form $N=P_3+P_3'$, where P_k, P_k' are numbers having at most k prime factors. In this connection, it should be mentioned that Atle Selberg announced [Proc. Internat. Congress Math. Cambridge, Mass., 1950, v. 1, Amer. Math. Soc., Providence, R.I., 1952, pp. 286–292; MR **13**, 438] the stronger result that $N=P_2+P_3'$. See also the related papers by Y. Wang [Acta Math. Sinica **6** (1956), 500–513, 565–582; MR **20** #4530; Sci. Record (N.S.) **1** (1957), 9–12] which use a combination of the methods of Kuhn [Norske Vid. Selsk. Forh. Trondheim **14** (1941), no. 39, 145–148; MR **8**, 503], Buhštab [Mat. Sb. N.S. **4(46)** (1938), 375–387; Dokl. Akad. Nauk SSSR **29** (1940), 544–548; MR **2**, 348] and Selberg.

The author employs the lower bound sieve method of Selberg given in the above paper and in a later one [11te Skand. Matematikerkongress, Trondheim, 1949, Tanum, Oslo, 1952, pp. 13–22; MR **14**, 726]. He follows a

suggestion of Selberg in taking $\lambda_p=1$, $\rho_1=1$ and

$$\rho_d=-\sum_{[d_1,\,d_2]=d/p}\lambda_{pd_1}\lambda_{pd_2},$$

where p is the largest prime divisor of $d>1$ and $[d_1, d_2]$ is the least common multiple of d_1 and d_2. Let the number of integers among $a_1, \cdots, a_N$ which are divisible by d be given by $N/f(d)+R_d$, where $f(n)$ is multiplicative, and let $f_1(n)$ be the completely multiplicative function which agrees with $f(n)$ at the primes. The author studies the function

$$(*)\qquad \varepsilon(z_p)=1-\sum_{\nu_p}\frac{\mu^2(\nu_p)}{f_1(\nu_p)}\cdot\prod_{p_1<p}\left\{1-\frac{1}{f(p_1)}\right\},$$

where $z_p=(z/p)^{\frac{1}{2}}$ and $\nu_p\leq z_p$ has as its largest prime divisor a number less than p. For the Goldbach problem in which $a_n=n(N-n)$, we have $f(p)=p/2$ if $p\nmid N$ and $f(p)=p$ if $p|N$. In this case, the author obtains the estimate $\varepsilon(z_p)=O(1/\log^3 z)$ if $p\leq z^{1/(8\log\log z)}$ and, in the other case, the estimate

$$\varepsilon(z_p)=\frac{1}{2\pi}e^{\omega-u}\int_{-\sqrt{\log z}}^{\sqrt{\log z}}\frac{e^{iut+\Psi(t)}}{(it-1)^3}dt+O\left(\frac{(\log\log z)^4}{e^u\sqrt{\log z}}\right),$$

where $u=(\log z_p)/\log p$, $\omega=2\sum_{n=1}^{\infty}1/(n!n)$ and $\Psi(t)=-ie\int_0^1(1-ix)^{-1}e^{-itx}dx$. The proof of this estimate depends on approximating the sum in (*) by

$$\frac{1}{2\pi i}\int_{a-it}^{a+it}\frac{z_p^s}{s}\prod_{p_1<p}\left\{1+\frac{1}{p_1^sf_1(p_1)}\right\}ds.$$

On moving the path of integration to the left and altering the product, one finds that $\varepsilon(z_p)$ is given, approximately, by the expression

$$\frac{1}{2\pi i}\int_L\frac{z_p^s}{s}\prod_{p_1<p}\left\{1+\frac{1}{p_1^{s+1}}\right\}^2ds,$$

since $f_1(p_1)=p_1/2$ with only a finite number of exceptions. Since $\prod_p(1+1/p^s)=\zeta(s)/\zeta(2s)$, the author is able to estimate $\prod_{p<y}(1+1/p^s)$ in terms of $\zeta(s)/\zeta(2s)$ and y. This is accomplished with the aid of another integral involving $\log\zeta(s+w)/\zeta(2s+2w)$ by shifting the path to the left again and using the fact that $\zeta(s)$ has no zeros in a region of the type

$$1-\lambda\{\log(|t|+e)\log\log(|t|+e^2)\}^{-3/4};$$

such a region results from the estimation of exponential sums as given by I. M. Vinogradov and his followers. Additional calculations now provide the formula for $\varepsilon(z_p)$.

Selberg's method now enables the author to conclude that for even N the number of integers of the type $n(N-n)$ with $1\leq n\leq N$ which are divisible by none of the consecutive primes $p_1, \cdots, p_r\leq z^{\xi}<p_{r+1}$ with $z=N^{1-\varepsilon}$ and $\xi=1/3.2$ has a lower bound

$$c\frac{N}{\log^2 N}\prod_{p|N}\left\{1+\frac{1}{p-1}\right\}\cdot\prod_{p\nmid N}\left\{1-\frac{1}{(p-1)^2}\right\}$$

for some $c>0$. His final result $N=P_3+P_3'$ follows at once. *L. Schoenfeld* (East Pittsburgh, Pa.)

3836:

Vinogradov, A. I. Application of $\zeta(s)$ to the sieve of Eratosthenes. Mat. Sb. N.S. **41** (83) (1957), 49–80; correction, 415–416. (Russian)

This paper is devoted to a proof of the theorem that every sufficiently large even number is representable as $p_1p_2p_3+p_4p_5p_6$, where $p_1, \cdots, p_6$ are primes. The corrigendum makes some substantial changes in the paper; even with these, the exposition does not seem to be altogether clear. *H. Davenport* (Cambridge, England)

3837:

Carlitz, L. The singular series for a single square. Portugal. Math. **16** (1957), 7–10.

In the problem of representation of integers as the sum of s squares, the singular series plays a dominant role [for the bibliography, see, for instance, P. T. Bateman, Trans. Amer. Math. Soc. **71** (1951), 70–101; MR **13**, 111]. The author considers the singular series for $s=1$, which turns out to be a very special case of Siegel's main theorem on positive quadratic forms [Ann. of Math. (2) **36** (1935), 527–606; theorem 2]. *S. Ikehara* (Tokyo)

3838:

Tong, Kwang-chang. On Waring's problem. Advancement in Math. **3** (1957), 602–607. (Chinese. English summary)

The result of this paper is

$$G(n)<n(3\ln n+9),\quad \text{when } n=2^m,$$
$$G(n)<n(3\ln n+7),\quad \text{when } n\neq 2^m.$$

From the author's summary

3839:

Trost, E. Eine Bemerkung zum Waringschen Problem. Elem. Math. **13** (1958), 73–75.

Regarding the value of $g(k)$ in Waring's problem, it is well known that: For all $k\geq 6$, for which the inequality 1) $3^k-2^kA_k\leq 2^k-A_k-2$, $A_k=[(3/2)^k]$, holds, 2) $g(k)=2^k+A_k-2$. The author shows that the above result can be stated in the form: If $k\geq 6$ and there is a natural number x_k in the interval

$$3)\qquad \frac{3^k}{2^k}+\lambda_k\leq x_k<\frac{3^k}{2^k}+1,\ \lambda_k=\frac{(3/2)^k+1}{2^k-1},$$

then

$$g(k)=2^k+x_k-3.$$

It is further shown that (3) is true for infinitely many k. *H. Gupta* (Chandigarh)

3840:

Grosswald, Emil. Some theorems concerning partitions. Trans. Amer. Math. Soc. **89** (1958), 113–128.

Let q be an odd prime, and let $\{a\}=\{a_1, \cdots, a_m\}$ be a set of m distinct least positive residues mod q. Let $F(x)=\prod_{\nu\equiv\{a\}}(1-x^\nu)^{-1}=\sum_{n=0}^{\infty}p_n(q)x^n$, where $\nu\equiv\{a\}$ means that ν runs only through those integers which are congruent (mod q) to some a_j ($j=1, 2, \cdots, m$); consequently, $p_n(q)$ is the number of partitions of n into summands congruent to elements of $\{a\}$. The author proves that

$$(1)\quad F(x)=$$
$$(-q\log x)^d\exp\left\{\frac{\Lambda}{-\log x}+K(-\log x)+O(\log^2 x)\right\}$$

as $x\to 1$, and that

$$(2)\quad p_n(q)=$$
$$\frac{\omega}{q}\left(\frac{q\Lambda^{1/2}}{\lambda_n}\right)^{d+1}\exp\left(\frac{d(d+2)}{4\Lambda^{1/2}\lambda_n}\right)I_1(2\Lambda^{1/2}\lambda_n)(1+O(n^{-1}))$$

as $n\to\infty$, where

$$\omega=\prod_{j=1}^{m}\{(2\pi)^{-1/2}\Gamma(a_j/q)\},\qquad d=\tfrac{1}{2}\sum_{j=1}^{m}(a_j-\tfrac{1}{2}q),$$
$$K=(24q)^{-1}\sum_{j=1}^{m}(q^2-6a_jq+6a_j^2),\quad \Lambda=m\pi^2/(6q),$$
$$\lambda_n=(n-K)^{1/2},$$

and $I_1(x)$ denotes the Bessel function. There are corresponding results about partitions into summands congruent to the elements of $\{a\}$, no summand being repeated more than l times; and several corollaries. The results of

this paper generalise theorems of Lehner [Duke Math. J. 8 (1941), 631–655; MR 3, 166], Livingood [Amer. J. Math. 67 (1945), 194–208; MR 6, 259], Meinardus [Math. Z. 59 (1954), 388–398; MR 16, 17] and also, in some respects, theorems of Petersson [Abh. Deutsch. Akad. Wiss. Berlin Kl. Math. Allg. Nat. 1954, no. 2; Acta Math. 95 (1956), 57–110; MR 17, 129, 1057]. The author points out that, although he might have used a method based essentially on the theory of modular functions, this would not have proved altogether satisfactory for completely general sets $\{a\}$. Instead, he uses a saddle point method, similar in some respects to that used by Meinardus. To prove (1) the author considers $F(x)$ as a product of m factors, one corresponding to each a_j ($j=1, \cdots, m$), and to each factor he applies a Mellin transformation. The integrand in the resulting contour integral involves Riemann's and Hurwitz' zeta functions; with the aid of their functional equations the line of integration is moved, and (1) is obtained by means of the calculus of residues. To deduce (2) from (1), the author applies the circle method to $p_n(q)=(2\pi r^n)^{-1}\{\int_0^{2\pi} F(re^{i\theta})e^{-2\pi in\theta}d\theta\}$, $|r|<1$; the range of integration is split up by a certain Farey dissection; and the major contribution is shown to be derived from the arc centred at $\theta=0$. The application of a theorem by Hayman [J. Reine Angew. Math. 196 (1956), 67–95; MR 18, 293] completes the proof of (2). The details of all of the proofs are complicated, but the method and main steps are clearly described. *H. Halberstam* (London)

3841:

Klimov, N. I. Combination of elementary and analytic methods in the theory of numbers. Uspehi Mat. Nauk (N.S.) 13 (1958), no. 3 (81), 145–164. (Russian)

I. V. Čulanovskiĭ [Dokl. Akad. Nauk SSSR 63 (1948), 491–494; MR 10, 355] has applied the Selberg sieve method to find an upper bound for various prime-number-theoretic functions, such as the number $Z_{u_1\cdots u_{m-1}}(N)$ of primes $p\leq N$ such that also $p+u_1, \cdots, p+u_{m-1}$ are prime. His method depended on the estimation of the sum function of a certain multiplicative function; he effected this estimation in an elementary fashion. In the present paper the author generalizes Čulanovskiĭ's results by considering almost primes, in progressions, and lying in more general intervals. His method is similar to that of Čulanovskiĭ, except that the required sum function is now estimated analytically, with the help of the ζ-function; this leads to improved results in the cases considered by Čulanovskiĭ. A generalization and improvement is also obtained of the heretofore best-known upper bound, due to Ricci [Ann. Scuola Norm. Sup. Pisa 6 (1937), 71–116], for the number $\pi^{(2)}(N, D)$ of primes $p\leq N$ of the form $p=n^2+D$.

W. J. LeVeque (Göttingen)

3842:

Tanaka, Minoru. On the number of prime factors of integers. III. Jap. J. Math. 27 (1957), 103–127

The author proves the following theorem: Let $f_i(x)$, $1\leq i\leq k$, be a set of polynomials with integral coefficients. Denote by $v_i(p)$ the number of solutions of $f_i(x)\equiv 0 \pmod p$. Let further π_i, $1\leq i\leq k$, be a set of prime numbers. Assume that for every i, $1\leq i\leq k$, $\sum_{p\in\pi_i} v_i(p)/p$ diverges. Define

$$\omega_i(n)=\sum_{\substack{p\mid n\\ p\in\pi_i}} 1,\quad y_i(n)=\sum_{\substack{p\leq n\\ p\in\pi_i}} \frac{v_i(p)}{p},\quad u_i(n)=\frac{\omega_i(f_i(n))-y_i(n)}{(y_i(n))^{\frac{1}{2}}}.$$

Let E be a Jordan-measurable set in k-dimensional space. Denote by $A(x, E)$ the number of integers $1\leq n\leq x$ for which the point $\{u_i(n)\}$, $1\leq i\leq k$, belongs to E. Then

$$\lim \frac{A(x,E)}{x}=(2\pi)^{-k/2}\int_E \exp\left(-\tfrac{1}{2}\sum_{i=1}^{k} u_i^2\right)du_1\cdots du_k.$$

Special cases of this theorem have been proved by the author in two previous papers [same J. 25 (1955), 1–20; J. Math. Soc. Japan 9 (1957), 171–191; MR 18, 563; 19, 636], and the proof is similar to the one used in these papers. *P. Erdös* (Haifa)

3843:

Rényi, A. Representations for real numbers and their ergodic properties. Acta Math. Acad. Sci. Hungar. 8 (1957), 477–493.

An "f-expansion" of a real number x is the sequence of integers $\{\varepsilon_n\}$ defined as follows: $\varepsilon_0(x)=[x]$, $r_0(x)=(x)$; $\varepsilon_{n+1}(x)=[f^{-1}(r_n(x))]$, $r_{n+1}(x)=(f^{-1}(r_n(x)))$. Here $[z]$ and (z) denote the integral part and fractional part of z, respectively. A real number x is said to have an f-expansion for a particular function f if either $r_n(x)=0$ for some n, so that the f-expansion terminates with ε_n, or the sequence $\{F_n(x, 0)\}$ converges to x. $F_n(x, y)$ is defined as follows: $F_0(x, y)=y$, $F_n(x, y)=F_{n-1}(x, \varepsilon_{n-1}(x)+f(\varepsilon_n(x)+y))$.

The problem of determining for what functions f all real numbers x have an f-expansion had previously been considered for monotone decreasing f by Bissinger [Bull. Amer. Math. Soc. 50 (1944), 868–876; MR 6, 150] and for monotone increasing f by C. J. Everett [ibid. 52 (1946), 861–869; MR 8, 259]. The results of the two authors are proved here with somewhat less restrictive assumptions on the function f. The main result of the article is concerned with measure-theoretic properties of f-expansions. It can be summarized as follows: Under certain conditions on f (somewhat more restrictive than those required for the existence of an f-expansion), and for a function g which is L-integrable on (0, 1) and otherwise arbitrary,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} g(r_k(x))$$

exists for almost all x and is a finite expression depending only on f and g, but not on x. The limit can be written as $\int_0^1 g(x)h(x)dx$, where h depends only on f and is such that $v(E)=\int_E h(x)dx$ is invariant under the transformation $(f^{-1}(x))$. By suitable choice of g and f, known theorems about regular continued fraction expansions, decimal expansions, as well as similar theorems for more general f-expansions, can be obtained.

W. J. Thron (Boulder, Colo.)

3844:

Godwin, H. J. On quartic fields of signature one with small discriminant. Quart. J. Math. Oxford Ser. (2) 8 (1957), 214–222.

The author previously enumerated the totally real quartic fields of discriminant $\Delta<11664$ [J. London Math. Soc. 31 (1956), 478–485; MR 18, 565] and the totally complex quartic fields of discriminant $\Delta\leq 1458$ [Proc. Cambridge Philos. Soc. 53 (1957), 1–4; MR 18, 565]. In this paper he considers quartic fields which have two real and two complex conjugates. In one table he gives a generating polynomial for each such quartic field which has no subfield and for which $-3280<\Delta<0$. In a second table he enumerates the quartic fields which have a subfield and for which $-3280<\Delta<0$. The latter extends a table of Delone and Faddeev [Trudy Mat. Inst. Steklov

11 (1940); MR 2, 349] in which there were some misprints. No discriminant occurs twice in either table, but $\Delta=-1472$ and $\Delta=-1984$ occur in both tables.

R. Hull (Pacific Palisades, Calif.)

3845:

Rédei, Ladislaus. Über die algebraischzahlentheoretische Verallgemeinerung eines elementarzahlentheoretischen Satzes von Zsigmondy. Acta Sci. Math. Szeged **19** (1958), 98–126.

An algebraic integer ω is called an exceptional number ω_n (n a rational positive integer), if there is no prime ideal $\mathfrak{p}$ in $R(\omega)$ with $\mathfrak{p}|\omega^n-1$, $\mathfrak{p}\nmid\omega^m-1$ for $m=1, 2, \cdots, n-1$. The author easily deduces a criterion for exceptionality. He also gives a new proof of a theorem by Zsigmondy [Monatsh. Math. 3 (1892), 265–284], in which the rational ω_n are summed up; in a companion paper [#3915 below] this theorem is applied to group theory. Further, he determines all quadratic ω_n. In a footnote the author mentions a paper by Sachs [Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg **6** (1956/57), 223–259; MR **19**, 391], which, among many other results, gives a more general form of the criterion mentioned above.

C. G. Lekkerkerker (Amsterdam)

3846:

Witt, Ernst. p-Algebren und Pfaffsche Formen. Abh. Math. Sem. Univ. Hamburg **22** (1958), 308–315.

This quite concisely written paper deals with a significant interpretation of the p-component of the Brauer group of algebra classes over a field k of characteristic p. The author interprets, thus extending his previous results [J. Reine Angew. Math. **176** (1936), 126–140], this group in terms of formal Pfaffian forms over the complete (unramified) field with residue class field k modulo total and integral differentials and a single relation involving the analogue of the Frobenius automorphism on the module of Pfaffian forms.

O. F. G. Schilling (Chicago, Ill.)

3847:

Cronheim, Arno. Ein Funktionenkörper von Primzahlcharakteristik ohne Automorphismen. Math. Nachr. **18** (1958), 99–105.

Suppose that k is an algebraically closed field of finite characteristic p. The author proves that the algebraic function field $k(x, y)$ of functions of the variable x defined by the equation $y^q+y^{q-1}+x^{qh}(x+1)=0$, with $q\geqq3$ a p-power, $h\geqq q$, and $(h, q-1)=1$, has only the identity automorphism over k. The proof involves direct computation of details used to establish the general theorem that K/k has a finite number of automorphisms [H. L. Schmid, J. Reine Angew. Math. **179** (1938), 5–15; F. K. Schmidt, Math. Z. **45** (1939), 62–96].

O. F. G. Schilling (Chicago, Ill.)

3848:

Pedersen, Peder. On the expansion of π in a regular continued fraction. Nordisk Mat. Tidskr. **6** (1958), 57–68, 95.

Here the author continues the computation of the continued fraction expansion

$$a_0+\frac{1}{a_1}+\frac{1}{a_2}+\cdots$$

for π up to a_{200}. He uses the step-by-step method of Lehmer [Amer. Math. Monthly **45** (1938), 227–233; **46** (1939), 148–152], who computed the partial quotients up to a_{100}. The author also computes the approximants A_{n-1}/B_{n-1} and A_n/B_n at the end of each step, that is, $A_{115}, B_{115}, A_{116}, B_{116}, A_{132}, B_{132}, A_{133}, B_{133}, \cdots, A_{200}, B_{200}$. Two different checks are made on the values of A_{200}, B_{200}. Finally, as Lehmer did (for $n=100$), the author obtains approximations to Khintchine's and Lévy's constants for $n=10, 20, \cdots, 200$. *E. Frank* (Chicago, Ill.)

3849:

İçen, Orhan Ş. Eine Verallgemeinerung und Übertragung der Schneiderschen Algebraizitätskriterien ins p-adische mit Anwendung auf einen Transzendenzbeweis im p-adischen. J. Reine Angew. Math. **198** (1957), 28–55.

Let $f(z)$ be an analytic function in the usual or in the p-adic sense [F. Loonstra, Thesis, Univ. of Amsterdam, 1941; MR 7, 111]. $f(z)$ is said to be algebraic in the arithmetic sense if $f(z)$ satisfies an algebraic equation $\sum_{r=0}^{a}\sum_{s=0}^{b} a_{rs}z^rf(z)^s=0$ with coefficients a_{rs} algebraic over the rationals. Let $\{z_n\}$ be a sequence of distinct algebraic numbers, z_n of height h_n. $\{z_n\}$ is said to be a permissible sequence if: (1) the z_n's are of uniformly bounded degree; (2) there is a ζ (not necessarily algebraic) and a $c>0$ such that for sufficiently large n, $|z_n-\zeta|\leqq h_n^{-c}$, where "$|\ |$" is the p-adic valuation or the usual absolute value; and (3) if for some $A>0$ and for sufficiently large n, the sequence $\{\log h_n\}$ satisfies

$$\sum_{r=1}^{n} \underline{\log h_r}>nA\,\overline{\log h_{r+1}},$$

where

$$\underline{\log h_n}=\min_{1\leq r\leq n}\log h_r,\quad \overline{\log h_n}=\max_{r\geq n}\log h_r.$$

The author's principal result is the following generalization of a theorem of Schneider [Acta Math. **86** (1951), 57–70; MR **13**, 735]. Let $f(z)$ be an analytic function which is regular at $z=\zeta$. Let $\{z_n\}$ be a permissible sequence converging to ζ such that the points $f(z_n)$ are in the domain of definition of $f(z)$, and: (a) the $f(z_n)$'s are algebraic numbers of uniformly bounded degree; (b) $\log H_n=o(n\log h_n)$ for $n\to\infty$, where H_n is the height of the algebraic number $f(z_n)$. Then $f(z)$ is algebraic in the arithmetic sense.

Conversely, if $f(z)$ is algebraic in the arithmetic sense and if $\{z_n\}$ is a sequence of algebraic numbers of uniformly bounded degree converging to a regular point ζ of $f(z)$ such that the points $f(z_n)$ are in the domain of definition of $f(z)$, then the sequence $\{f(z_n)\}$ satisfies (a) and (b) above.

Further theorems of this type are proved and an application is given to the proof of the Mahler-Veldkamp theorem on the p-adic transcendence of α^β [K. Mahler, Compositio Math. **2** (1935), 259–275; G. R. Veldkamp, J. London Math. Soc. **15** (1940), 183–192; MR **2**, 149].

Morris Newman (Washington, D.C.)

3850:

İçen, Orhan Ş. Eine weitere Verallgemeinerung eines Schneiderschen Algebraizitätskriteriums. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **21** (1956), 155–187 (1957); Fehlerverzeichnis zur Arbeit, 261. (Turkish summary)

The author generalises a result of his thesis [reviewed above], using the same method and notation. Let $\{z_n\}$ be an infinite sequence of distinct (complex or p-adic) algebraic numbers of bounded degrees and heights h_n. Let there be a (complex or p-adic) ζ and a $c>0$ such that $|z_n-\zeta|\leqq h_n^{-c}$ for $n\geqq n_0$. Let $\{j_n\}$ be a sequence of positive integers such that

$$J_{n+1}=O(J_n)\text{ where } J_n=\sum_{r=1}^{n} j_r,$$

ring S, $S^n=0$, while $R^n\neq 0$, n a fixed natural number and let

$$\log h_{n+1}=O(J_n^{-1}\sum_{\nu=1}^{n} j_\nu \log h_\nu).$$

Let $f(z)$ be an analytic function of the (complex or p-adic) variable z regular in a neighbourhood of $z=\zeta$. Let all numbers

$$a_{nj}=\frac{1}{j!}f^{(j)}(z_n)\quad (j=0, 1, \cdots, j_n-1)$$

for each n be algebraic and lie in an extension of the rational field of bounded degree. Denote by B_n a positive integer such that all $B_n\cdot a_{nj}$ are algebraic integers, by H_{nj} the height of a_{nj}. Finally let

$$\log B_n=O(J_n \log h_n),\ \log H_{nj}=O(J_n \log h_n).$$

Then $f(z)$ is an algebraic function. A converse theorem also holds. This result leads to a new proof of the transcendency of the p-adic exponential function [K. Mahler, J. Reine Angew. Math. **169** (1931), 61–66; A. Günther, ibid. **192** (1953), 155–166; MR **15**, 604]. — The paper unfortunately is full of bad misprints.

K. Mahler (Manchester)

3851:

Ridout, D. The p-adic generalization of the Thue-Siegel-Roth theorem. Mathematika 5 (1958), 40–48.

Let $f(x)$ be a polynomial of degree $n\geqq 2$ with rational coefficients, and let $\zeta, \zeta_1, \cdots, \zeta_t$ be a real root, a p_1-adic root, $\cdots$, a p_t-adic root of $f(x)=0$, respectively; here $p_1, \cdots, p_t$ are distinct primes. Assume κ is a constant such that the inequality

$$\min\left(1, \left|\zeta-\frac{h}{q}\right|\right)\prod_{\tau=1}^{t}\min(1, |h-q\zeta_\tau|_{p_\tau})\leqq(\max(|h|, q))^{-\kappa}$$

has infinitely many solutions in integers h, q, where $(h, q)=1$ and $q>0$. Using Siegel's method of the proof of the Thue-Siegel theorem, the reviewer showed in 1932 [Math. Ann. **107** (1933), 691–730] that $\kappa<2\sqrt{n}$. The new method due to K. F. Roth [Mathematika 2 (1955), 1–20, 168; MR **17**, 242] enables the author to show that κ cannot be larger than 2, a result that is best possible.

K. Mahler (Manchester)

3852:

Macbeath, A. M.; and Rogers, C. A. A modified form of Siegel's mean value theorem. II. Proc. Cambridge Philos. Soc. **54** (1958), 322–326.

Λ is defined to be a discrete point set in real Euclidean n-space for which $\lim_{r\to\infty} N(r)/r=d$, where $N(r)$ is the number of points of Λ in a sphere of radius r with centre at the origin. If γ is an $n\times n$ matrix with determinant 1, $\|\gamma\|$ is defined to be $\sup_{|x|\leq 1}|\gamma x|$, x being a point in Euclidean n-space. Where $\rho(x)$ is a function which vanishes outside a bounded region, $\rho(\gamma\Lambda)$ is defined to be $\sum\rho(x)$, the sum to be taken over all x in $\gamma\Lambda$. In part I of this paper [same Proc. **51** (1955), 565–576; MR **17**, 241] the authors defined a measure $\mu(\gamma)$ over sets of matrices γ and proved that if $\int\rho(x)dx$ exists as a Riemann integral, then

$$\int_{\|\gamma\|\leq K}\rho(\gamma\Lambda)d\mu(\gamma)\Big/\int d\mu(\gamma)\to d\int\rho(x)dx,$$

as $K\to\infty$, where $\int\rho(x)dx$ is defined over the whole space. The present paper uses this result to establish the same limit if $\int\rho(x)dx$ is integrable in the Lebesgue sense.

D. Derry (Vancouver, B.C.)

COMMUTATIVE RINGS AND ALGEBRAS

See also 3871.

3853:

Edge, W. L. The geometry of an orthogonal group in six variables. Proc. London Math. Soc. (3) 8 (1958), 416–446.

The finite projective space of five dimensions over the Galois field with the marks 0, 1, −1 (mod. 3) consists of 364 points. In it are two kinds of non-singular quadric; the paper is interested in the type with 112 points which has no planes on it; the equation of Q may be written as $\sum x_i^2=0$ $(i=1, \cdots, 6)$. There are two batches of 126 points off Q, points k and points l, according as the $\sum$ is 1 or −1. The author studies the simple group G^* of $2^7\cdot 3^6\cdot 5\cdot 7$ projectivities which leave Q invariant, do not transpose the categories k and l, have determinant 1 and permute the 126 points of each batch evenly. The group was known to L. E. Dickson, who, however, did not give any geometrical application. The group is transitive on the 567 "positive" simplexes, that is, self-polar simplexes of Q whose vertices are all k points. Led by his geometrical considerations, the author studies the Sylow 2-groups (of order 2^7) of G^* and the subgroups S_3. This paper contains many other noteworthy results in the geometry considered, such as the 540 null systems which reciprocate Q into itself and the 5184 heptahedra that are circumscribed to Q.

O. Bottema (Delft)

3854:

Endler, Otto. Kennzeichnung abelscher Körpererweiterungen vom Grad p^m und von vorgegebenem Erweiterungstypus über einem Teilkörper des Grundkörpers. I. Kennzeichnung im Grundkörper. Math. Nachr. 17 (1958), 73–92.

Let k be a field of characteristic p, and K be an abelian extension, of degree a power of p, over k. With a subfield k_0 of k, over which k is Galois, the paper studies (as in Hasse's work [Abh. Deutsch. Akad. Wiss. Berlin, Math.-Nat. Kl. **1947**, no. 8; MR **11**, 155] for the case where $(K:k)$ is not divisible by p) the case when K is Galois over k_0 and has, indeed, as the Galois group over k_0 a prescribed extension of the Galois group of K/k. The study is made in terms of Witt's [J. Reine Angew. Math. **176** (1937), 126–140] theory (which characterizes K by Witt vectors in k) and by means of cohomology of the Galois group of k/k_0 in the group of Witt vectors in k.

T. Nakayama (Nagoya)

3855:

Sce, Michele. Osservazioni sulle serie di potenze nei moduli quadratici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 220–225.

The subject of this note is finite ordered quadratic modules with unit; that is, modules with unit in which every element satisfies a quadratic equation. The author proves that in every such module of even order, over a field of characteristic different from 2, the series of powers annihilates a power of a generalized laplacian. This generalizes a result of Fueter not available to the reviewer [R. Fueter, Comment. Math. Helv. 7 (1934–35), 307–330].

R. L. San Soucie (Buffalo, N.Y.)

3856:

Rédei, Ladislaus. Die einstufig Nicht-Zeroringe und Verallgemeinerungen. Abh. Math. Sem. Univ. Hamburg 22 (1958), 201–214.

A ring R having the property that for each proper sub-

$\geqq 2$, is said to be a not n-nilpotent ring of degree one. Such a ring may be generated by n elements, so if r denotes the least number of generators of R, then $1\leqq r\leqq n$. The author, considering the cases $r=1$ and $r=n$, proves that then R is finite and determines all such rings. For example, the class consisting of (i) the rings defined by a generating element ρ and the equations $\rho^{2n}=0$, $p\rho^{2n-1}=0$, $\cdots$, $p^{n-1}\rho^{n+1}=0$, $p^n\rho^n=0$, $p^k\rho=0$ (k a natural number and p a prime), (ii) the homomorphs R of the preceding rings when $R^n\neq 0$, and (iii) the rational integers mod p^m ($m=1, \cdots, n$), is precisely the class of not n-nilpotent rings of degree one with one generator. The problem, as attacked by the author, is basically the determination of the structure of certain ideals in the ring of polynomials over the rational integers.

W. E. Deskins (East Lansing, Mich.)

3857:

Nagata, Masayoshi. On the derived normal rings of Noetherian integral domains. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 29 (1955), 293–303.

L'auteur donne des démonstrations des résultats suivants, dûs à Mori [Bull. Kyoto Gakugei Univ. Ser. B, no. 9 (1956), 1–5; no. 10 (1957), 1–5; no. 11 (1957), 1–7; MR 19, 938]: la clôture intégrale A' d'un anneau noethérien intègre A est un anneau de Krull (=anneau normal, endlich diskret Hauptordnung); si, de plus, A est de dimension 1 ou 2, A' est noethérien. Les démonstrations données ici sont plus simples que celles de Mori; elles considèrent d'abord le cas où A est local (A' est alors semi-local).

{Dans la démonstration de la proposition 2 (p. 295, lignes 3, 4 et 5), il y a de gênantes fautes d'impression: au lieu de $\mathfrak{a}b$ il faut lire ab (3 fois)}.

P. Samuel (Clermont-Ferrand)

3858:

***Séminaire M. Krasner de la Faculté des Sciences de Paris, 1953/1954. Théorie des corps valués. Vol. 1 [Exposés nos. 1-4].** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956. 52+25+46+39 pp. (multigraphiées)

It is well known that every valuation of a field (or ring, or group) induces a topological structure of metric type on the field, in which the triangle axiom must be replaced by (*) $d(x, z)\leqq\max[d(x, y), d(y, z)]$. For a general valuation over an abstract field, in the sense of Krull, the distance is a function with values in an arbitrary ordered abelian group, and its corresponding space is called by the author an ultrametroid space. The present exposition begins with the study of ultrametroid spaces (Exp. no. 1). Exp. no. 2 studies valued groups and rings; no. 3, the valuations of fields and their extensions; no. 4, the structure defined by the so-called multiplicative valuations of a field. We shall sketch the contents of each.

Exp. no. 1. An ultrametric space E is a space in which is defined a distance, with values in a multiplicative pseudo-group of non-negative reals, for which the triangle axiom is replaced by the axiom (*). If the multiplicative pseudo-group is an arbitrary ordered pseudo-group, the space is called ultrametroid. Since all the results obtained for the ultrametric case can be translated at once to the ultrametroid, most of the present exposition deals with the former. For a systematic study of ultrametric spaces, the author introduces the concept of semireal number, which is a specialization to the reals of the complete monoid of Kurepa [Thèse, Paris, 1935]. To each real r correspond three semireal numbers: r^-, r^+, r^0. The first two are the set of all monotone increasing or decreasing sequences, respectively, of reals whose limits are r. The third is the set of sequences with only a finite number of terms distinct from r. A circle $C(a; \rho)$ with center a and radius ρ, where ρ is a semireal non-negative number of type $-$ or 0, is the set of all points x of E such that $d(a, x)\leqq\rho$. The set D_ρ of all circles $C(a; \rho)$ of E is called a divisor of E. If x and y are points of the same circle of D_ρ, set $|D_\rho|=\sup d(x, y)$; it is called the valuation corresponding to D_ρ. The set of all divisors of E is a complete lattice. Among other properties are proved the following: E is a totally disconnected space; if $\mathscr{D}$ is a set of divisors D of E, the set $\mathscr{F}$ of their corresponding quotient spaces E/D is a projective system; if $\bar{R}$ is a set of non-negative reals such that $\inf \bar{R}=0^+$, the projective limit $\bar{E}$ of E/D_{r-}, $r\in\bar{R}$, is a complete ultrametric space independent of $\bar{R}$.

Exp. no. 2. A metric [metroid] group is a set G with structures of both a group and a metric [metroid] space, such that, for $a, x, y\in G$, $d(ax, ay)=d(xa, ya)=d(x, y)$. If e is the unit element of G, the function $|x|=d(x, e)$ is called a pseudovaluation [a pseudofiltration]. If the distance is ultrametric [ultrametroid], i.e., if the triangle axiom is replaced by (*), the pseudovaluation is called a valuation [a filtration]. Necessary and sufficient conditions are given for a non-discrete ultrametroid group to be compact. A ring A is called an ultrametroid ring if it is an ultrametroid space and its additive group is an ultrametroid group which admits its own elements as operators, i.e., $d(ax, ay):d(x, y)=\|\lambda\|$, where $\|\lambda\|$ is a constant. Then it is shown that $|xy|=\mu|x|\,|y|$. If $\mu=1$, the ring is called a filtered ring. The author gives the principal properties of filtered rings and some criteria for a ring to be a valuation ring.

Exp. no. 3. This lecture is an application of results of the preceding two to obtain the classical results of the theory of valuated fields. Its contents are: Valuated fields, equivalence of valuations; completion of valuated fields; complete fields as projective limits of rings; examples of valuated fields; projectivization of valuated fields; extension of a valuation.

Exp. no. 4. Contents: Multiplicative congruences; skeletons and fieldoids. Let $|\cdots|$ be a valuation of a field k. The function

$$(**)\qquad |x|_m=|x-1|/\max(1, |x|)$$

is a valuation of the multiplicative group k^* of k. If Π_ρ is a multiplicative divisor, i.e., a divisor of k^* relative to this valuation, and if H_ρ is the set of classes mod Π_ρ, one can define a product for these classes, since if $\bar{x}_\rho=C(x; \rho|x|)$ and $\bar{y}_\rho=C(y; \rho|y|)$ are two of them, then $\bar{x}_\rho\bar{y}_\rho=C(xy; \rho|xy|)=\overline{xy}_\rho$. The author points out two ways to define an addition in H_ρ. 1) One can define $\bar{x}_\rho+\bar{y}_\rho$ as the set of all classes of H_ρ contained in the set-theoretical sum of $\bar{x}_\rho$ and $\bar{y}_\rho$. This sum is not generally an element of H_ρ. 2) In the more important case of $s=H_1^-$ one can define an addition for which the following holds. A) If $\bar{x}$ can be added to $\bar{y}$, then $\bar{x}+\bar{y}\in s$. B) If $\bar{x}+\bar{y}$ is defined, and if the set-theoretical sum of $\bar{x}$ and $\bar{y}$ is a unique class $\bar{z}$ mod Π, then $\bar{x}+\bar{y}=\bar{z}$. C) The operation inverse to addition is single-valued when it exists. D) $\bar{0}$ is a zero for this addition. E) There exists the opposite, $-\bar{x}$, of every $\bar{x}\in s$ and $\bar{0}\in\bar{x}+(-\bar{x})$. The author analyzes the algebraic structures corresponding to both definitions of addition. A fieldoid ("corpoide") is a set Q provided with two operations, addition and multiplication, and a relation, addibility, such that: A) Q is a pseudogroup relative to multiplication. B) Addition is defined only between addible pairs. The set R of $x\in Q$ addible with 1 is an

abelian group relative to addition. If $x \in Q$ is addible with $y \in R$, $y \neq 0$, then $x \in R$. C) Multiplication is bilaterally distributive relative to addibility and to addition. It is proved that R is a field, called the field of the fieldoid. The multiplicative group R^* of R is an invariant subgroup of the multiplicative group Q^* of Q. The group $\mathfrak{M}=Q^*/R^*$ is called the group of the fieldoid. It is proved that s is a fieldoid, called the skeleton of the valuated field k. It is shown that Q^* is a Schreier extension of R^* by $\mathfrak{M}$, and that given a field R and a group $\mathfrak{M}$ there exist Schreier extensions of R^* by $\mathfrak{M}$ which are fieldoids with R and $\mathfrak{M}$ as field and group, respectively. If Q is an overfieldoid of q, R and $\mathfrak{M}$ the field and group of Q, and r and $\mathfrak{m}$ those of q, then $(R:r)$ is called the field-degree and $(\mathfrak{M}:\mathfrak{m})$ the group-degree of the extension Q/q. The fieldoid-degree $(Q:q)$ is defined and it is proved that $(Q:q)=(R:r)(\mathfrak{M}:\mathfrak{m})$. A fieldoid extension Q/q is called "étalé" if $(R:r)>1$, completely "étalé" if $(R:r)>1$ and $(\mathfrak{M}:\mathfrak{m})=1$, "étagé" if $(\mathfrak{M}:\mathfrak{m})>1$, and completely "étagé" if $(\mathfrak{M}:\mathfrak{m})>1$ and $(R:r)=1$. *P. Abellanas* (Madrid)

3859:

✱Séminaire M. Krasner de la Faculté des Sciences de Paris, 1953/1954. Théorie des corps valués. Vol. 2 [Exposé no. 5]. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956. 131 pp. (multigraphiées)

Contents: 1) Modules without torsion. Their rational closure and their extensions. 2) Linear varieties in the space $L^q \times \mathfrak{M}$. 3) Polynomials over a fieldoid. 4) Algebraic and transcendental extensions of a fieldoid. 5) Structure of the pure transcendental extensions of rational type. 6) Simple algebraic extensions. 7) Completely "étagé" algebraic extensions. 8) Separability theory. 9) Applications to valued extensions.

The object of this exposition is to develop an elementary theory of the extensions of commutative fieldoids without torsion. It is known [see the preceding review] that a fieldoid is completely characterized by its corresponding field and group. A fieldoid is said to be without torsion if its group is without torsion.

It is proved that a fieldoid extension Q/q is algebraic if and only if its field extension R/r is algebraic and its group extension $\mathfrak{M}/\mathfrak{m}$ is rational. An element x of $\mathfrak{M}$ is rational relative to $\mathfrak{m}$ if there exists an element $\lambda \in L$, where L is the operator ring of $\mathfrak{m}$, such that $\lambda x \in \mathfrak{m}$. Otherwise x is said to be transcendental over $\mathfrak{m}$. A fieldoid is algebraically closed if and only if its field is algebraically closed and its group is rationally closed. If $[Q:q]$, $[R:r]$ and $[\mathfrak{M}:\mathfrak{m}]$ are the transcendence degrees of the fieldoid, field and group extensions respectively, then they satisfy $[Q:q]=[R:r]+[\mathfrak{M}:\mathfrak{m}]$. A transcendental extension Q/q is called homogeneous if $[\mathfrak{M}:\mathfrak{m}]=0$ or $[R:r]=0$; otherwise it is called heterogeneous. All the algebraic closures of a fieldoid are isomorphic to one another.

Let X be a set, which may be infinite, of symbols distinct from one another, and let q be a fieldoid with field r and group $\mathfrak{m}$. Let $\hat{\mathfrak{m}}$ denote the rational closure of $\mathfrak{m}$, i.e., the set of all solutions of the equations $lx=a$, where $l \in L$ and $a \in \mathfrak{m}$. Every function $\pi=\pi(x)$ defined over X with values in $\hat{\mathfrak{m}}$ is called an $\hat{\mathfrak{m}}$-gradient of X. If $\bar{X} \subseteq X$, the restriction of $\pi(x)$ to $\bar{X}$ will be called the $\hat{\mathfrak{m}}$-gradient of $\bar{X}$ induced by π and denoted by $\bar{\pi}=\pi_{\bar{X}}$. Let f be a formal power sum on $\bar{X}=\{x_1, \cdots, x_s\} \subset X$, with coefficients in q. Let $\mathscr{L}$ be the ring of rational integers; to every term $a_{i_1,\cdots,i_s} x_1^{i_1} \cdots x_s^{i_s}$ of f one can associate the point $M_{i_1,\cdots,i_s}=(i_1, \cdots, i_s, \omega(a_{i_1,\cdots,i_s}))$ of $\mathscr{L}^s \times \mathfrak{m}$, where $\omega(a)=ar^*$ is the element of $\mathfrak{m}$ that contains a. The linear variety $N(f)$ of $\mathscr{L}^s \times \mathfrak{m}$ generated by the set $M(f)$ of the points $M_{i_1,\cdots,i_s}$ corresponding to all the terms of f is called the Newton variety of f. A hyperplane $x_1\alpha_1+\cdots+x_s\alpha_s+x\alpha=c$ is called vertical or non-vertical according as $\alpha=0$ or $\alpha\neq 0$. The vector $(-\alpha^{-1}\alpha_1, \cdots, -\alpha^{-1}\alpha_s)$ is called the gradient of the hyperplane. A linear variety is called vertical if every hyperplane containing it is vertical; otherwise it is called non-vertical. It is proved that $f(x_1, \cdots, x_s)$ is a polynomial if and only if its Newton variety is not vertical. The polynomial f is said to admit gradient π if there exists a hyperplane of gradient π that contains $N(f)$. The set of all polynomials admitting gradient π over q will be denoted by $q_\pi[x]$. The quotients fg^{-1} of polynomials of gradient π over q are called rational x-fractions of gradient π over q, and will be denoted by $q_\pi(x)$. It is proved that $q_\pi(x)$ is a fieldoid which is, with trivial identifications, an overfieldoid of q, and that the extension $q_\pi(x)/q$ is pure transcendental of rational type. Conversely, if Q/q is a pure transcendental extension of rational type, it is isomorphic to some extension $q_\pi(x)/q$ for a suitable gradient π.

The simple algebraic extensions $q(\alpha)$ of a fieldoid q are characterized by the same theorem as the simple algebraic field extensions; i.e., an algebraic extension Q of a fieldoid q is simple if and only if it is isomorphic to $q_\pi[x]/I_\alpha$, where $\pi=-\omega(\alpha)$ and I_α is the prime ideal of $q_\pi[x]$ formed by all the polynomials for which α is a zero.

An element α of an extension Q of a fieldoid q is called a kummerian element if a power α^d $(d\neq 0)$ of α belongs to q. The least d with this property is called the exponent of α. Q/q will be called a kummerian extension if there exists a set A of kummerian elements such that $Q=q(A)$. If Q/q is completely "étagé" [see the preceding review], every element of Q is a kummerian element. If g is an abelian group, denote by $g^{(d)}$ the subgroup of the dth powers of the elements of g and by g_d the group $g/g^{(d)}$. Let Q be a completely "étagé" extension of q and $Q(d)$ the multiplicative group of all the elements of Q whose exponents divide d; then $Q(d)^{(d)} \subseteq q^*$ and $q^{*(d)} \subset Q(d)^{(d)}$. Let $\Gamma_d(Q/q)=Q(d)^{(d)}/q^{*(d)} \subseteq q^*{}_d$. A parametrized family $\Gamma=\{\Gamma_d\}_{d=1,2,\cdots}$ of subgroups Γ_d of the corresponding $q^*{}_d$ is called a kummerian characteristic of q if it satisfies the following conditions. 1) $d|d'$ implies that $p_{d,d'} \cdot \Gamma_{d'} \subseteq \Gamma_d$, where $p_{d,d'}$ is the application $aq^{*(d')} \rightarrow aq^{*(d)}$ of $q^*{}_{d'}$ onto $q^*{}_d$. 2) $\Gamma_d \cap r^*{}_d=\{1_d\}$. 3) If $d|d'$ and $\delta=d'd^{-1}$, the restriction of $\eta_{d',d}$ to Γ_d is an isomorphism of this group onto $\Gamma_{d'} \cap q^*{}_d(\delta)$, where $\eta_{d',d}$ is the homomorphism $aq^{*(d)} \rightarrow a^\delta q^{*(d')}$ of $q^*{}_d$ into $q^*{}_{d'}$. The structure of the completely "étagé" extensions Q/q is given by the following theorem: Two completely "étagé" algebraic extensions of a fieldoid q are q-isomorphic if and only if they have the same kummerian characteristic. The following existence theorem is also proved: If Γ is a given kummerian characteristic of q, there exist completely "étagé" algebraic extensions Q of q such that $\Gamma(Q/q)=\Gamma$. These theorems solve completely the problem of the structure of the algebraic extensions of a fieldoid, since every algebraic extension Q/q is the product of the two extensions Q/q^R, which is a completely "étagé" extension, and q^R/q, which is characterized by the field extension R/r.

If Q is an extension of q, and $\alpha \in Q$, then α is a separable element over q if and only if the field extension and the group extension corresponding to the extension $q(\alpha)/q$ are separable and p-separable, respectively. One can at once extend the theorem on kummerian characteristics to inseparable extensions Q/q and obtain the following: Two

inseparable extensions of q are isomorphic fieldoids if and only if they have the same characteristic (defined analogously to the kummerian case).

Brief applications are made to extensions of valuations.
P. Abellanas (Madrid)

3860:

Ribenboim, P. Anneaux normaux réels à caractère fini. Summa Brasil. Math. 3 (1956), 213–253.

Using methods and procedures of proof first developed systematically by W. Krull [J. Reine Angew. Math. **167** (1932), 160–196], the author generalizes theorems of Artin and van der Waerden concerning ideal theory in integrally closed rings. Notation: K a given field; v a valuation on K with valuation ring A_v and value group Γ_v; A a subring of K; P a prime ideal in A; A_P the associated quotient ring; and R^+ the ordered group of real numbers. According to Krull [loc. cit.], a ring A is integrally closed in K if and only if it is the intersection $\bigcap_v A_v$ of suitable valuation rings; any family $\{A_v\}$ of valuation rings is termed a defining family of A if $A=\bigcap_v A_v$; such $A_v \supset A$ are called associated to A. Furthermore, a valuation v associated to A is called essential if $A_v=A_P$ for some prime ideal $P\subset A$. Application of the methods and results indicated above implies easily that (i) every defining family Ω contains essential valuations, and (ii) if A has a defining family of essential valuations, then all $P\subset A$ are maximal if and only if $\Gamma_v\subseteq R^+$ for all v associated to A. For the rings with these properties it is assumed that, apart from $\Gamma_v\subseteq R^+$, for every $x\in K$ there are only a finite number of v's in Ω for which $v(x)\neq 0$ ("finite type"). After preliminaries of this type the author turns to the problem of describing modules M and fractional ideals of K with respect to an integrally closed subring A. As in Krull's work, he associates to each M two functions on Ω: (i) f_M defined by $f_M(v)=\inf\{v(x), x\in M\}$ ($-\infty$ being admitted in the obvious manner), and (ii) $s_M=e$ if there exists an $x\in M$ with $v(x)=f_M(v)$, and u otherwise. The problem initiated by Krull is to characterize the A-modules M by their associated functions f and s, with values in $R^+\cup(-\infty)$ and $\{e, u\}$. The results of Artin and van der Waerden have already shown that a unique characterization is not possible (roughly speaking, if the Kroneckerian dimension is larger than one). To each M there is associated a closure $\bar{M}$ defined by $\bar{M}=\{x\in K,\ v(x)\geqq f_M(v)\}$ if $s_M(v)=e$, and $\bar{M}=\{x\in K, v(x)>f_M(v)\}$ if $s_M(v)=u$. Then closed modules $\bar{M}$ are uniquely defined by the values of f and s on Ω, as is to be expected. The author states the effects of the operations $M\to\bar{M}$ and $M\to(f, s)$ (lexicographical ordering on the additive set of all functions on Ω to $R^+\cup\{e, u\}$ being used to define a sup and inf, e.g., $(f'', s'')=(f, s)\vee(f', s')$ if $M''=M\cap M'$, etc.). Furthermore, for each M, the closure $\bar{M}=\bigcap_{v\in\Omega} MA_v$. Significant for the proofs of assertions of this type is the approximation theorem for rings of finite type: If $v_1, \cdots, v_m\in\Omega$ and $\alpha_i\in\Gamma_{v_i}$ are given, then there exists $x\in K$ with $v_i(x)=\alpha_i$ and $v(x)\geqq 0$ for $v\in\Omega-\{v_1, \cdots, v_m\}$. Since the valuations of Ω are not assumed to be discrete, changes have to be made in order to obtain decomposition theorems for closed modules $\bar{M}$ and ideals. Thus, the author introduces, for positive real ρ, the concept of essential power $M^\rho=\{x\in K|v(x)\geqq\rho f_M(v)\}$ if $s_M(v)=e$, and $M^\rho=\{x\in K|v(x)>\rho f_M(v)\}$ otherwise, for closed M; furthermore, $\{\bar{M}\}$ is topologized by setting $V_{F,n}(\bar{M})=\{M'\in\{\bar{M}\}, |f_{\bar{M}}(v)-f_{M'}(v)|\leqq 1/n,\ s_{M'}(v)=s_{\bar{M}}(v)$ for all $v\in F$, a given finite subset of $\Omega\}$. Then it follows easily that in each neighborhood of $\bar{M}$ there is an essential power product with integral exponents of ideals which are "close" to minimal prime ideals of A. The second part of the paper deals with the simplifications arising if the above approximation theorem is strengthened: In addition to the α_i given previously, elements $x_i\in K$ are given; then there exists $x\in K$, with $v_i(x-x_i)\geqq\alpha_i$ and $v(x)\geqq 0$, $v\in\Omega-\{v_1, \cdots, v_m\}$. This theorem is proved, as obviously may be expected, by assuming that every associated v of A has $\Gamma_v\subseteq R^+$ and that A has a defining family of essential valuations (finite type being understood). Rather immediate consequences of these assumptions are that all $P\subset M$ are maximal, the A-submodules of K are uniquely characterized by the pairs (f, s), and the operations of inf and sup in the monoid of modules, ordered by $M|M'$ if $M'=M\cdot I$ for an integral ideal $I\subseteq A$, can be described by means of the functions f and s.
O. F. G. Schilling (Chicago, Ill.)

3861:

Ribenboim, P. Corps maximaux et complets par des valuations de Krull. Math. Z. 69 (1958), 466–479.

This paper is a continuation of the author's previous amplifications of W. Krull's basic paper on valuation theory [J. Reine Angew. Math. **167** (1932), 160–196]. Now, the concepts of completeness and maximal completeness of a field K with respect to a valuation w are examined. Krull showed [loc. cit.] that a maximally complete field K (having no extension with identical value group and residue class field for the given valuation) is necessarily complete (certain systems of congruences having solutions in K). Kaplansky [Duke Math. J. **9** (1942), 303–321; **12** (1945), 243–248; MR **3**, 264; **7**, 3] characterized maximal fields in terms of pseudo-convergent sequences. Here, the author introduces the concept of distinguished pseudo-convergent sequence (a_τ) of elements $a_\tau\in K$; τ in a set of well-ordered subscripts T. It is required that: (i) (a_τ) is pseudo-convergent, i.e., $w(a_\tau-a_{\tau'})<w(a_{\tau'}-a_{\tau''})$ for $\tau<\tau'<\tau''$ in T; (ii) the breadth of (a_τ), i.e., the ideal $\{x\in K, w(x)>\gamma_\tau=w(a_\tau-a_{\tau'}), \tau\in T, \tau<\tau'\}$ is a prime ideal p distinct from the maximal ideal belonging to w, and (iii) all elements a_τ lie in the valuation ring of w. Then K is complete with respect to w if and only if every distinguished pseudo-convergent sequence of elements in K has a pseudo-limit in K, i.e., if there is an element $a\in K$ for which $w(a-a_\tau)=w(a_{\tau'}-a_\tau)$, $\tau<\tau'$, for every $\tau\in T$. The proof of this theorem involves arguments familiar from the work of Krull and Kaplansky. Furthermore, the author corrects an assertion of the reviewer and proves a generalization of a result by F. K. Schmidt: A field in which Hensel's lemma holds for two valuations of the same rank whose valuation rings have no limit prime ideals is either algebraically closed or has an algebraically closed residue class field.
O. F. G. Schilling (Chicago, Ill.)

3862:

Nilov, G. N. Contribution to the investigation of the roots of a cubic equation. Kabardin. Gos. Ped. Inst. Uč. Zap. 12 (1957), 28. (Russian)

An extremely concise proof is given for the fact that if two roots of the cubic equation $x^3+px+q=0$ are equal, then $q^2/4+p^3/27=0$.
From the introduction

3863:

Breitenberger, Ernst. Partial fractions. Amer. Math. Monthly 64 (1957), 654–657.

In this paper the author is concerned with the decomposition of genuine rational functions into partial fractions.

Consider the fraction $f(x)/g(x)$. Split g into two coprime factors ϕ_1 and ψ_1. Let $f/g=\phi_1^*/\phi_1+\psi_1^*/\psi_1$, where g is a rational function of degree n in x. Then, comparing coefficients of like powers of x in $f=\phi_1^*\psi_1+\psi_1^*\phi_1$, we get equations which are easily written in the matrix form:

$$R(\psi_1, \phi_1)\binom{B}{A}=K$$

where R is a matrix involving the coefficients of ψ_1 and ϕ_1 and A, B, K are the one-column matrices containing the coefficients of ϕ_1^*, ψ_1^* and f. The process is repeated for ψ_1^*/ψ_1.

The case where g is a power of a polynomial is also considered. *H. Gupta* (Chandigarh)

3864:

Nagata, Masayoshi. An example of a normal local ring which is analytically reducible. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 31 (1958), 83–85.

Soient k un corps de caractéristique $\neq 2$, x, y, z des indéterminées sur k, et $\sum_{i\geq 1} a_i x^i$ une série formelle transcendante sur $k(x)$; posons

$$z_1=z, \quad z_{n+1}=x^{-n}(z-(y+\textstyle\sum_{j<n} a_j x^j)^2),$$

$$\mathfrak{r}=k[x, y, z_1, \cdots, z_n, \cdots]_{(x,y,z_1,\cdots,z_n,\cdots)}, \quad \mathfrak{o}=\mathfrak{r}[W]/(W^2-z_1)$$

(où W est une nouvelle indéterminée). Alors $\mathfrak{o}$ est un anneau local intègre et intégralement clos de dimension 2, et un module de type fini sur l'anneau local $\mathfrak{r}$ qui est régulier (on montre que $\mathfrak{r}$ est noethérien en montrant que tout idéal premier de $\mathfrak{r}$ est de type fini). Dans le complété de $\mathfrak{o}$ l'idéal (0) est intersection de deux idéaux premiers. *P. Samuel* (Clermont-Ferrand)

ALGEBRAIC GEOMETRY

3865:

Burniat, Pol. Superficie algebriche regolari di genere geometrico $p_g\geq 4$ qualunque e di genere lineare $p^{(1)}=2p_g-3, 2p_g-2, \cdots, 8p_g+7$. I. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 24 (1958), 276–281, 404–409.

L'A. démontre ici par des exemples l'existence de surfaces algébriques régulières ayant le genre géométrique $p_g\geq 4$ et pour lesquelles le genre linéaire $p^{(1)}$ peut prendre toutes les valeurs comprises entre $2p_g-3$ et $8p_g+7$; quelques doutes restent pour $p^{(1)}=8p_g+2$, dans le cas où p_g est pair. La méthode employée par l'auteur dans la construction des exemples dont il est question est la même qu'il a déjà employée autrefois, c'est-à-dire la construction de certains plans quadruples abéliens qui sont définis à partir de trois courbes de diramation convenables E_1, E_2, E_3. Il n'est pas possible d'exposer ici les détails de la construction; ils rentrent dans les propriétés générales de ce type de plans quadruples, propriétés que l'auteur répète au commencement en se rapportant à ses travaux antérieurs sur ce sujet [voir par exemple mêmes Atti (8) 16 (1954), 326–331; MR 16, 615]. *E. G. Togliatti* (Genoa)

3866:

Gallarati, Dionisio. Sulle varietà di Fano con curve sezioni canoniche. Rend. Mat. e Appl. (5) 16 (1957), 315–327.

A Fano variety is an M_r^{2p-2} normal in S_{r+p-2} whose general curve section is a canonical curve of genus p. Roth [Proc. Cambridge Philos. Soc. 46 (1950), 419–428; MR 12, 355] studied those for which $r=3$, 4 on which the only hypersurfaces are complete sections; the present paper is devoted to those on which the only hypersurfaces are multiples of a single system (itself a submultiple of a hyperplane section.) It is shown that the only M_r^{2p-2}, $r\geq 4$, satisfying this condition are the Veronesian K_5^{32} of quadrics in S_5 and its hyperplane section M_4^{32}, for which $p=17$. Several examples are also given of M_r^{2p-2} not satisfying these conditions, but whose general hyperplane sections do so; some of these are special cases of well-known varieties, but of more interest are: The projective model M_4^{24} of all cubics in S_4 which trace a fixed cubic surface on a fixed plane; the projective model M_3^{36} of all quartic surfaces in S_3 which trace a fixed quartic curve on a fixed plane. *P. Du Val* (London)

3867:

Matsusaka, T. On a theorem of Torelli. Amer. J. Math. 80 (1958), 784–800.

L'auteur démontre le théorème suivant, déjà démontré dans le cas classique par R. Torelli et A. Andreotti, et dans le cas abstrait par A. Weil: soient C, C' deux courbes complètes non-singulières, J, J' leurs jacobiennes, f, f' les applications canoniques de C, C' dans J, J', Θ et Θ' les diviseurs canoniques sur J, J'; s'il existe un isomorphisme i de J sur J' tel que $i(\Theta)$ soit numériquement équivalent à Θ', alors il existe une transformation birationnelle h de C sur C' telle que $f'\circ h=\pm i\circ f+$cte (le signe $\pm$ est arbitraire dans le cas où C et C' sont hyperelliptiques, uniquement déterminé par i dans le cas contraire; une fois le signe déterminé, h et la constante sont uniques).

La démonstration sépare le cas où C et C' sont hyperelliptiques, et celui où l'une des courbes n'est pas hyperelliptique. Elle comprend divers lemmes sur l'application canonique f d'une courbe C dans sa jacobienne J. Le plus expressif montre que, en notant W^r la sous-variété $f(C)+\cdots+f(C)$ (r fois) de J, les points simples w de W^r sont ceux pour lesquels il existe un diviseur positif $\mathfrak{m}$ de degré r sur C tel que $w=S(f(\mathfrak{m}))$ et que $l(\mathfrak{m})=1$. Un autre lemme montre qu'une application rationnelle séparable d'une courbe complète non singulière sur une autre de même genre est birationnelle. L'auteur généralise aussi un résultat dû à Humbert et Castelnuovo: une variété complète non singulière en codimension 1 ne porte pas de familles algébriques non triviales de faisceaux (="pencil") irrationnels de diviseurs.

P. Samuel (Clermont-Ferrand)

3868:

d'Orgeval, Bernard. Sur une inégalité de Commessatti. Bul. Inst. Politehn. Iaşi (N.S.) 3 (1957), no. 1–2, 11–14. (Russian and Romanian summaries)

If q_2, q_3 are the superficial and three-dimensional irregularities of an algebraic threefold V_3, Commessatti proved transcendentally [Atti Accad. Lincei Rend. (5) 22 (1913), sem. 2, 316–321] that $q_3\geq q_2-3$, unless V_3 contains either an irrational pencil of surfaces or an irrational congruence of curves. The present paper gives a proof of the same theorem on classical algebro-geometric lines. *P. Du Val* (London)

3869:

Godeaux, Lucien. Sur le système jacobien d'un système linéaire de surfaces algébriques. Mathesis 67 (1958), 5–7.

A partir du calcul direct du jacobien de quatre formes, l'A. montre que si les surfaces algébriques d'un systéme

triplement infini passent par un point avec plan tangent commun, ce point est double pour le jacobien du système. Il en résulte que les surfaces fondamentales de seconde espèce (non rencontrées en des points variables par les surfaces du système) appartiennent comme composantes doubles au système jacobien. *B. d'Orgeval* (Dijon)

3870:

✱**van der Waerden, B. L. On the definition of rational equivalence of cycles on a variety.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 545–549. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

È il resoconto di uno scambio epistolare tra F. Severi, P. Samuel e l'autore originato da una discussione che seguì la lezione di Severi al Simposio di Geometria Algebrica [lo stesso vol., 529–541; MR **20** #5206]. Al termine della discussione Samuel propose una definizione di "equivalenza razionale" sopra una varietà algebrica, che però poco dopo egli stesso riconobbe inadatta, perchè, essendo una equivalenza più fine di quella di Severi, non permetteva di definire l'intersezione di due classi di equivalenza. L'A., dopo aver esaminato alcune precedenti definizioni di Severi, si mostra propenso ad adottare la definizione di equivalenza razionale che si trova nell'opera di F. Severi, "Serie, sistemi di equivalenza e corrispondenze algebriche sulle varietà algebriche", v. 1 [Edizioni Cremonese, Roma, 1942; MR **10**, 206]. Egli elabora infine un insieme di definizioni, equivalenti a quelle di Severi ma valide nell'ambito dell'algebra astratta, le quali consentono di estendere la teoria dell'equivalenza razionale ad una varietà algebrica definita sopra un campo qualunque. *M. Rosati* (Roma)

3871:

✱**Séminaire H. Cartan et C. Chevalley, 8e année: 1955/1956. Géométrie algébrique.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956. ii+173 pp. (polycopiées)

Contents: J. Lafon: 1. Quotient rings. Integral elements. Theorems of Krull. 2. Nötherian rings. — H. Cartan: 3. Affine algebraic varieties. — M. Lazard: 4. Affine algebras. — C. Chevalley: 5. Schemes, I. 6. Schemes, II. — C. Chevalley and H. Cartan: 7. Normal schemes; morphisms; constructible sets. — H. Cartan: 8. Morphisms and constructible sets. — C. Chevalley: 9. Correspondences, I. 10. Correspondences, II. 11. The main theorem of Zariski, I. 12. The main theorem of Zariski, II. — P. Cartier: 13. Derivations in a field. 14. Regular extensions. — P. Cartier and C. Chevalley: 15. Extensions of the ground field by schemes. — R. Godement: 16. Simple localities, I. 17. Simple localities, II. 18. n-adic topologies. 19. Analytic properties of the localities.

In some questions of algebraic geometry one finds that it would be legitimate to identify the concepts of point of an algebraic variety and of local ring at that point. The guiding idea of the present work consists in developing the foundations of algebraic geometry starting from this identification.

Let K be a field and L a finite extension of K. Let A be an affine algebra, i.e., a subalgebra of L considered as an algebra over K. A locality [=spot] of L is a local ring corresponding to a prime ideal of an affine algebra. A locality M' dominates a locality M when $M' \supset M$ and $r(M') \supset r(M)$, where $r(M)$ and $r(M')$ are the radicals of M and M', respectively. The locality M is said to be related to the locality M' when there exists another locality dominating both. A locality M is a specialization of another locality N when $N = M_{\mathfrak{p}}$, where $\mathfrak{p}$ is a prime ideal of M. An affine scheme is the set of localities which are local rings of all the prime ideals of an affine algebra, called its definition algebra A scheme [=model] is a set S of localities of L such that: i) S is the union of a finite number of affine schemes; ii) two distinct localities of S are not related. Let H be a vector space over K of finite dimension >0, contained in L; denote by A_u ($u \in H$, $u \neq 0$) the affine algebra generated by $u^{-1}H$ over K. If S_u is the affine scheme corresponding to A_u, the union of all S_u, $u \in H$, $u \neq 0$, is a scheme, called the projective scheme defined by H. A scheme is complete if every locality of L is related to at least one locality of the scheme. Every projective scheme is complete. The topology of Zariski is defined for the affine schemes and is extended to the topology $\mathcal{T}$ of the set $\mathcal{L}$ of all the localities of L. If S is a scheme, one defines the topology of Zariski over S as that induced by $\mathcal{T}$. The topologization of the schemes permits the definition of the concept of irreducibility; it is proved that every scheme is irreducible, since there exists a unique locality Q such that every other locality of the scheme is a specialization of Q. Q is a field, called the field of quotients of the scheme. A scheme S' dominates another scheme S when every locality of S' dominates a locality of S. If S and S' are two schemes of L, there exists a unique scheme S'' of L dominating both S and S' and dominated by any other such scheme. A scheme is called normal when all its localities are normal, i.e., integrally closed. Let S be a scheme of L, F its field of quotients, $F' \subset L$ an algebraic extension of F. Then there exists a unique scheme S' of L such that: a) S' is normal, dominates S, and has F' as its field of quotients; b) it is minimal relative to this condition. S' is called the normal derived scheme of S in F'. When $F = F'$, S' is called the normal derived scheme of S. All this, corresponding to the preliminary work relative to the definition of derived normal model by Zariski [Amer. J. Math. **61** (1939), 249–294], gives rise to lectures 5, 6 and half of 7.

Let S and S' be two schemes relative to the same field K with F and F' as the corresponding fields of quotients. A morphism (f, φ) of S onto S' is defined by: 1) a continuous correspondence $f\colon S \to S'$ (i.e., such that $M \subset M' \Rightarrow f(M) \subset f(M')$); 2) a ring homomorphism $\varphi\colon f(F) \to F$ such that, for every locality $M \in S$, $\varphi[f(M)] \subset M$ and $\varphi[r(f(M))] \subset r(M)$. If S dominates S', if $f\colon S \to S'$ is the application of domination, and if $\varphi\colon F' \to F$ is the canonical injection of F' in F, the morphism (f, φ) is called the morphism of domination. Every morphism (f, φ) of S onto S' decomposes into a morphism (f'', φ'') of S onto S'', where S'' is the scheme induced by S' over the set of specializations of $f(F)$ and φ'' is injective, and a morphism (f', φ') of S'' onto S', where φ' is an epimorphism. There is a one to one correspondence between the morphisms of an affine scheme S onto another affine scheme S' and the homomorphisms of their corresponding algebras. A correspondence (S'', f, f') between two schemes S and S' consists of a scheme S'' and two morphisms f and f' of S'' onto S and S' respectively. Two correspondences (S'', f, f') and (S_1'', f_1, f_1') are considered equal when there exists an isomorphism j of S'' onto S_1'' such that $f = j \circ f_1$, $f' = j \circ f_1'$. A locality $M \in S$ and a locality $M' \in S'$ correspond to each other when there exists a locality $M'' \in S''$ such that $f(M'') = M$ and $f'(M'') = M'$. Every locality corresponding to a given

locality of S is a specialization of a finite number of localities of S'. Bounds are given for the dimension of the transform of a given locality under a correspondence. Let Γ be a non-degenerate correspondence between S and S', $\Gamma=(S'', f, f')$; one can assume that S'' dominates S and S' and that f and f' are applications of domination. Let S_0'' be the union of S and S', h the application of domination of S'' into S_0'', and f_0 that of S_0'' into S. A locality M of S is fundamental for Γ if: a) M is fundamental for S_0''; or b) there exists a locality in $f_0^{-1}(M)$ which is not a specialization of any locality of $h(S'')\cap f_0^{-1}(M)$. Bounds are given for the dimension of the image of a fundamental locality. The locality M' is said to dominate regularly the locality M when: a) The tr. deg. of the quotient field of M equals that of the quotient field of M'; b) $\dim (M)=\dim (M')$; c) There is no locality $\neq M'$ dominating M and of which M' is a specialization. Then the main theorem of Zariski [Trans. Amer. Math. Soc. **53** (1943), 490–542; MR **5**, 11] can be stated in the following form: If M' dominates M regularly, then $M'=M^*_{m^*}$, where M^* is the integral closure of M in M' and m^* is a maximal ideal of M^*. All this corresponds to the last part of lecture 7 and to lectures 8–12.

Lecture 13 studies the derivations defined in a ring or field, their extensions to an overfield, and the characterizations, by means of derivations, of the separable extensions of a field.

If L_1 and L_2 are two extensions of K, $\mathcal{T}=(E, u_1, u_2)$ is called an extension of K compounded with L_1 and L_2 when E is an extension of K generated by two algebraically disjoint subextensions L_1' and L_2' and u_i $(i=1, 2)$ are K-isomorphisms of L_i onto L_i'. An extension L^* of K is called primary when the algebraic closure of K in L^* is purely inseparable. A separable primary extension L/K is called regular. Lecture 14 gives some characterizations of such extensions. Lecture 15 studies the effects of extension of the field of constants on the localities.

In lecture 16 the characteristic function of a graded module is defined and it is proved that if M is a nötherian local ring, q a primary ideal of M, and E an M-module of finite type, the polynomial function $\chi_E(q; n)$ has a degree independent of q, called the height of the M-module E. In the case $E=M$ is a locality, it is shown that the height of M equals the length of the maximal chain of prime ideals in M, as well as the number of elements of the minimal basis of a certain primary ideal of M. Then, in lecture 17, a regular local ring is defined, as usual, by the condition that its radical possess a basis with a number of elements equal to its height, and the principal properties of such local rings are proved, including Zariski's criterion [ibid. **62** (1947), 1–52; MR **9**, 99].

Lecture 18 studies Zariski's m-adic rings [Summa Brasil. Math. **1** (1946), 169–195; MR **9**, 265] and their application to semi-local rings and complete rings, giving a very concise proof of Cohen's theorem in the case of equal characteristics [Trans. Amer. Math. Soc. **59** (1946), 54–106; MR **7**, 509] and of Hensel's lemma. Some properties of simple localities are proved in lecture 19. In the first four lectures is collected the algebraic material necessary for the subsequent exposition.

P. Abellanas (Madrid)

3872:

✱Zariski, Oscar. Introduction to the problem of minimal models in the theory of algebraic surfaces. Publications of the Mathematical Society of Japan, no. 4. The Mathematical Society of Japan, Tokyo, 1958. vii+89 pp.

This publication, though deceptively small in size, is heavy in thought. Its content is algebraic geometry written by one of the algebraic geometers' algebraic geometers. Among the prerequisites for the reader are a thorough knowledge of the theories of Chow-van der Waerden points of algebraic varieties, specialization of cycles and local rings. Granted the essence of these results the author presents a significant modern approach to classical problems of the theory of algebraic surfaces, and their proper generalization to fields of characteristic p. In the first part of the memoir the basic properties of rational transformations of algebraic varieties over an algebraically closed field are treated. The discussion is quite general and reflects the author's previous work on algebraic correspondences, and the full proof of Bertini's theorem. The second part is devoted to the study of exceptional curves on an algebraic surface without singularities, the existence of such models of an algebraic function field depending on the decisive results of the author and his pupil Abhyankar. Here results known for characteristic zero and first systematically presented in the author's earlier book [Algebraic surfaces, Ergeb. Math. ihrer Grenzgeb., Springer, Berlin, 1935] are developed in detail. For example, as an important tool for later proofs the factorization theorem of anti-regular transformations into quadratic transformations is proved (local considerations). An anti-regular transformation T is a birational transformation of an algebraic variety V such that T^{-1} is regular at every point Q' of $T\{Q\}$, the total transform of Q on V, i.e., the local ring of Q' contains the local ring of Q. Also, the behavior of the canonical system of a surface with respect to birational transformations is discussed. The second part includes results (proved to every detail) such as the fact that the irreducible components of an exceptional curve of the first kind (every point of the curve is regular — local rings — and the curve is the total transform of a simple point by some T) are non-singular curves. In the third part of the treatise the author sketches the theory of minimal models, i.e., the lower bounds in the ordered set of non-singular projective models F, F' of an algebraic function field of 2 variables, $F<F'$ if the birational transform F' of F is anti-regular (at every point). The theorem of Néron and Severi concerning the behavior of the base number of algebraic divisors relative to birational transformations is used to prove that each birational class $\{F\}$ satisfies the minimum condition for the relation $<$. Here connections with Serre's results on algebraic sheaves may be made. The author indicates how his new approach ultimately leads to a proof of Castelnuovo's celebrated theorem that a subfield of a rational function field $k(x, y)$, k algebraically closed, over which the latter is separable, must be rational. Detailed proofs of this and other generalizations of classical theorems are announced for future publication.

O. F. G. Schilling (Chicago, Ill.)

3873:

Zariski, Oscar. The problem of minimal models in the theory of algebraic surfaces. Amer. J. Math. **80** (1958), 146–184.

A birational class B of non-singular varieties is the set of non-singular projective varieties which are birationally equivalent to a variety. More precisely, in the set of pairs (V, P) of non-singular varieties V which are birationally equivalent to a variety V_0 and a generic point P of V, we introduce an equivalence relation such that (V, P) and (V', P') are equivalent if and only if the birational correspondence between V and V' such that P and P' are

corresponding generic points is a biregular transformation between V and V'; then $\boldsymbol{B}$ is the set of equivalence classes. $\boldsymbol{B}$ is a partially ordered set: $(V, P) \prec (V', P')$ if the birational correspondence T from V' onto V such that P' and P are corresponding generic points is a regular mapping.

A relatively minimal model of $\boldsymbol{B}$ is a minimal member of $\boldsymbol{B}$. If the smallest member exists, then it is called the minimal model of $\boldsymbol{B}$.

The purpose of this article is to prove the following theorems. Fundamental theorem A: If a birational class $\boldsymbol{B}$ of non-singular surfaces contains no minimal model, then $\boldsymbol{B}$ contains a ruled surface. Fundamental theorem B: If a non-singular surface F carries an irreducible exceptional curve E of the second kind, then F is birationally equivalent to a ruled surface, and if, furthermore, the self-intersection number (E^2) of E is (strictly) positive, then F is a rational surface. (A curve E on a surface F is said to be an exceptional curve if it is the total transform of a simple point on a birationally equivalent surface; it is said to be of the first kind if it can be the anti-regular total transform of a simple point; otherwise, it is said to be of the second kind. An exceptional curve (cycle) of the first kind has self-intersection number -1, while one of the second kind has non-negative self-intersection number: see the author's monograph on the problem of minimal models [#3872 above]). Fundamental theorem A was proved (not completely rigorously) by Castelnuovo and Enriques [Ann. Mat. Pura Appl. (3) **6** (1901), 165–168] in the classical case; their proof is not applicable to the abstract case.

In order to prove these fundamental theorems, the author first proves: If $\boldsymbol{B}$ is a birational class of non-singular varieties (of arbitrary dimension) and if $V \in \mathbf{B}$, then there exists a relatively minimal model $V' \in \boldsymbol{B}$ such that $V' \prec V$. For the proof, the author uses the theorem of Néron-Severi [see A. Néron, Bull. Soc. Math. France, **80** (1952), 101–166; MR **15**, 151; and the reviewer adds here that another proof using sheaf theory was given by H. Matsumura (not published yet)]. A simple proof for the case of surfaces based on properties of exceptional curves of the first kind was given by the author [forthcoming in Mem. Coll. Sci. Univ. Kyoto].

By the properties of exceptional curves of the second kind, the author shows that fundamental theorem A is a consequence of fundamental theorem B. As for the proof of fundamental theorem B, though the case where $(E^2)=0$ is rather easy, the case $(E^2)>0$ involves deep observation on rationality of surfaces. Preliminary results are contained in the author's monograph reviewed above, whose pp. 78–84 contains a sketch of this article.

M. Nagata (Cambridge, Mass.)

3874:

Leung, K. T. Ein Satz über lokal normale Varietäten. Math. Ann. **134** (1958), 232–236.

The author gives a very complicated proof of the following fact {which is well known and which is an immediate consequence of the fact that if R is a Noetherian normal ring then $R = \bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$ where $\mathfrak{p}$ runs over all prime ideals of rank 1}: Let P be a normal point on an algebraic variety V with a generic point $(x)=(x_1, \cdots, x_n)$ over a field of definition k. If $f \in k(x)$ is finite at P (i.e., f has no pole at P), then f can be expressed in the form $g(x)/h(x)$ with $g(x)$, $h(x) \in k[x]$ and such that $h(P) \neq 0$.

M. Nagata (Cambridge, Mass.)

3875:

Nagata, Masayoshi. Existence theorems for nonprojective complete algebraic varieties. Illinois J. Math. **2** (1958), 490–498.

It is proved that for any algebraic variety of dimension >1 there exists a complete normal birationally equivalent abstract variety which cannot be embedded in a projective space. Furthermore, one can construct a complete nonsingular nonprojective variety birationally equivalent (over the prime field) to a given projective space of dimension >2. The first part reduces easily to the case of the projective plane, after which the basic idea is the following: If V is a projective variety, W a subvariety contained in an open subset V' of V, $T: V' \to \bar{V}$ a regular surjective birational map which is biregular on $V'-W$, then $V-W+T(W)$ is a complete abstract variety, nonprojective if things are done properly.

M. Rosenlicht (Evanston, Ill.)

3876:

Marchionna, Ermanno. Sul teorema di Riemann-Roch relativo alle varietà algebriche. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 396–404.

The theorem of Riemann-Roch for algebraic varieties has been proved in a topological form by various authors using the modern theory of stacks. The theorem has recently been translated into the language of classical algebraic geometry by Hodge [J. London Math. Soc. **30** (1955), 291–296; MR **17**, 1133]. The main point of the present paper is to prove that the Hodge version of the Riemann-Roch theorem can be obtained independently of the theory of stacks, by using some theorems of Kodaira. In a certain sense the method is analogous to the one used by Italian geometers in proving the Riemann-Roch theorem for surfaces.

The paper also contains observations on the connection of the irregularity of an algebraic variety with the deficiency and superabundance of certain of its linear systems.

D. Pedoe (Khartoum)

3877:

Kirby, D. Isolated intersections of a set of n primals in n-space. J. London Math. Soc. **33** (1958), 185–196.

The present paper gives an algebraic construction for certain spaces introduced geometrically by the reviewer [Ann. Mat. Pura Appl. (4) **41** (1956), 113–138; MR **18**, 334] in connection with an isolated intersection of n primals in n-dimensional projective space, and extends further the investigation of local properties concerning such a set of primals.

Let K be a field of characteristic zero, and denote by Q the ring of formal power series $K\{x_1, \cdots, x_n\}$, by $\mathfrak{m}$ its maximal ideal $(x_1, \cdots, x_n)$, and by $\mathfrak{q}$ any $\mathfrak{m}$-primary ideal of Q. If m is an integer such that $\mathfrak{q} \supseteq \mathfrak{m}^{m+1}$, the equations $X_{i_1 \cdots i_n} = x_1^{i_1} \cdots x_n^{i_n}$ $(i_1 + \cdots + i_n \leq m)$ represent a Veronese variety, $V_n^{(m)}$, in the affine space A where the $X_{i_1 \cdots i_n}$ are coordinates. The substitution defined by the above equations maps the K-module $K[x_1, \cdots, x_n]$ mod $(x_1, \cdots, x_n)^{m+1}$ isomorphically onto the K-module of polynomials of the first degree in the $X_{i_1 \cdots i_n}$, and thus gives a subspace $S^{(m)}(\mathfrak{q})$ of A — containing the origin P of the X-space — as a representative of $\mathfrak{q}/\mathfrak{m}^{m+1}$. It is shown that the length of $\mathfrak{q}$ exceeds the dimension of $S^{(m)}(\mathfrak{q})$ by one, that the differences s_j in the dimensions of the intersections of $S^{(m)}(\mathfrak{q})$ with the j- and the $(j-1)$-osculating spaces of $V_n^{(m)}$ at P $(j=1, \cdots, m)$ are invariant over any regular transformation of the x_i, and

that the space $S^{(m)}(\mathfrak{q})$ is in a sense independent of the integer m and can be regarded as a model of the Macaulay inverse system of $\mathfrak{q}$.

When K is the complex field and $\mathfrak{q}$ is generated by n elements $f_1, \cdots, f_n$ of $K[x_1, \cdots, x_n]$, such that the corresponding primals $f_i(x_1, \cdots, x_n)=0$ have only a finite number of points in common, then $S^{(m)}(\mathfrak{q})$ does not differ from the space introduced by the reviewer, and leads to a simple connection between the geometric and algebraic definitions of the multiplicity of intersection of the n primals at the origin O of the x-space.

More generally, if K denotes again any field of characteristic zero, and if $\mathfrak{q}$ is generated by n elements $f_1, \cdots, f_n$ of Q, then the multiplicity of intersection of the corresponding analytic primals at O is given by the length of $\mathfrak{q}=(f_1, \cdots, f_n)$. Moreover, if $\rho+1$ is the exponent of $\mathfrak{q}$, i.e., if $\mathfrak{q}\supseteq\mathfrak{m}^{\rho+1}$ but $\mathfrak{q}\not\supseteq\mathfrak{m}^{\rho}$, then the smallest osculating space of $V_n^{(m)}$ at P which contains $S^{(m)}(\mathfrak{q})$ is precisely the ρ-osculating space, and $S^{(m)}(\mathfrak{q})$ intersects the i-osculating space of $V_n^{(m)}$ at P in a space of dimension d_i, where $d_i+1=\text{length}(\mathfrak{q}+\mathfrak{m}^{i+1})$ $(i=0, 1, \cdots, \rho)$, so that $d_\rho+1$ is the above multiplicity of intersection; further: $d_{i+1}>d_i$ $(i=0, \cdots, \rho-1)$, $d_0=0$ (and so $d_\rho\geq\rho$), $d_{r-1}+d_{\rho-r}\geq d_\rho-1$ $(r=1, 2, \cdots, \rho)$. In particular, when the tangent cones at O to the primals $f_1, \cdots, f_n$ have no generator in common, then $\rho=t_1+t_2+\cdots+t_n-n$, $d_{r-1}+d_{\rho-r}=d_\rho-1$ $(r=1, 2, \cdots, \rho)$, where t_i denotes the multiplicity of the primal f_i at O.

B. Segre (Rome)

3878:

Godeaux, Lucien. Sur le contact de deux surfaces algébriques le long d'une courbe. Publ. Sci. Univ. Alger. Sér. A 4 (1957), 103–106.

3879:

Trempont, J. Sur une transformation birationnelle involutive du plan. Mathesis 66 (1957), 302–305.

Etude de la transformation birationnelle qui fait correspondre aux droites du plan les quintiques ayant six points doubles A_1, A_2, A_3, A_4, B_1, B_2 non situés sur une conique et conservant le faisceau des coniques par les A_i. Cette transformation se représente sur la surface cubique, image des cubiques planes passant aux A et B, par une homographie biaxiale dont un des axes est l'image de B_1B_2; il en résulte que les courbes fondamentales associées à B_1 et B_2 sont respectivement les coniques (A, B_2) et (A, B_1); les points unis de la transformation sont ceux de la droite B_1B_2 et les trois points S_k diagonaux du quadrangle des A; les courbes fondamentales associées aux A_i sont les coniques définies par les cinq points-base autres que A_i. Les six points-base ne peuvent être quelconques, la conique AB_i étant tangente à la droite B_1B_2 à l'autre point B. Sur une conique la transformation engendre une involution dont le point de Frégier est le pôle de B_1B_2 par rapport à la conique; le lieu de ce pôle est la conique définie par les B et les S.

B. d'Orgeval (Dijon)

3880:

Deaux, R.; et Clodic, M. Équations du sixième degré à racines groupées en ternes involutifs. Mathesis 66 (1957), 129–138.

Les racines d'une équation du 6° degré déterminent quinze couples et l'on cherche s'il peut exister parmi ces couples des ternes formant trois couples de points conjugués dans une involution de Möbius. Etude algébrique. Il peut exister de 1 à 6 ternes de cette nature. S'il y en a 2, ils ont un couple en commun. S'il y en a 3, deux quelconques n'ont pas de couples communs et les six racines forment deux cycles d'une homographie d'ordre 3. S'il y en a 4, il existe un terne ayant un couple commun avec chacun des trois autres, eux-mêmes sans couples communs 2 à 2; les racines forment alors un cycle d'une homographie du sixième ordre. S'il y en a six, ils se groupent en trois couples tels que les ternes de chacun de ces couples aient un couple commun. Etude des involutions associées. Représentation géométrique.

B. d'Orgeval (Dijon)

3881:

Togliatti, Eugenio G. Sulla matrice caratteristica d'una trasformazione Cremoniana tra piani. Univ. e Politec. Torino. Rend. Sem. Mat. 16 (1956–57), 361–370.

In any Cremona transformation between two planes, if $m, \lambda_1, \cdots, \lambda_h$ are the order of a linear system of curves in one plane and its base multiplicities in the base points of the transformation, and $m', \lambda_1', \cdots, \lambda_h'$ are the analogous characters of the corresponding system in the other plane, then the transformation from one set of characters to the other is linear, the rows of the matrix of coefficients being the analogous characters of the hemaloids and of the fundamental curves with change of sign in all but the first column; these satisfy classically certain quadratic and linear conditions. The author points out that if we replace the two sets of characters by the vectors $m, i\lambda_1, \cdots, i\lambda_h$; $m', i\lambda_1', \cdots, i\lambda_h'$, the quadratic conditions mean that the transformation is orthogonal, and the linear conditions mean that it leaves invariant the vector $(3, i, \cdots, i)$. The rest of the paper is devoted to the block analysis of the matrices corresponding to two transformations and their resultant, where they have some but not all of their base points and fundamental curves in common.

P. Du Val (London)

3882:

van der Waerden, B. L. Ueber André Weils Neubegründung der algebraischen Geometrie. Abh. Math. Sem. Univ. Hamburg 22 (1958), 158–170.

A. Weil introduced in his "Foundations of algebraic geometry" [Amer. Math. Soc. Colloq. Publ., New York, 1946; MR 9, 303] the concept of regular extension $k(x)/k$, where (x) denotes a finite set of quantities over k. He proved the important characterization of such fields as precisely those fields $k(x)$ which are separably generated over k, and for which k is algebraically closed in $k(x)$. The author presents a slightly more arithmetic proof of this theorem, which exploits the fact that a separable extension of $k(x)$ implies, by the theorem on the primitive elements, $k(x)=k(y_1, \cdots, y_r, y_{r+1})$, with y_{r+1} separable over the purely transcendental field $k(y_1, \cdots, y_r)$. This remark implies a reduction to the factorization of the irreducible polynomial $f(Y; y_1, \cdots, y_r)$ over $k(y_1, \cdots, y_r)[Y]$ with the root y_{r+1} for coefficient extensions. Furthermore, it is shown how this method may be used to simplify proofs concerning the factorization of the prime ideal belonging to a regular set (x) in a polynomial ring upon extension of the coefficient field.

O. F. G. Schilling (Chicago, Ill.)

LINEAR ALGEBRA

See also 4221.

3883:

✱Graeub, Werner. Lineare Algebra. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. 97. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. xi+219 pp. Paperbound: DM 35.70; clothbound: DM 39.00.

This textbook developed from the author's lectures at the University of Zürich, 1954–56. There are 11 chapters, the scope of which may be seen from the following specific comments.

A general remark is that the author limits himself to finite dimensional spaces throughout and never tells his reader that much of what he covers works in other situations.

In Ch. 1, the definition of finite dimensional (there is a finite maximal number of linearly independent vectors) is not followed up properly, with the result that the coordinate space F_n is not proved to be finite dimensional. Ch. 2, on linear transformations, is, all-in-all, quite satisfactory. The point of the section on dual pairs of spaces is not very clear since it merely repeats the material on the conjugate space in the first chapter. No attention is given to computations, so that the beginner might find it difficult to compute, say, the inverse of a matrix on the basis of this treatment. Since the book is obviously aimed at students with very little background, some concrete examples would be a real help.

In Ch. 3, a "determinant function" on an n-dimensional space is an n-multilinear functional which vanishes whenever its arguments are linearly dependent. This latter condition rather than the usual, and simple, "alternating" makes the existence of a non-trivial one clumsy. The algebra of the symmetric group is carefully avoided so that the proof that a transformation and its transpose have the same determinant, established by means of a dual determinant function, is quite involved. The insistence on determinant functions for doing everything leads to complications that are unnecessary. The fact that the characteristic function $|\lambda I-A|$ is actually a polynomial is very hard, and we never find out that the coefficients are sums of principal minors.

The main point of Ch. 4 (Oriented linear spaces) is a proof that the set of real matrices of positive determinant is arc-wise connected. This would have been easier after covering euclidean spaces, since rotations are used.

In Ch. 5, on multilinear algebra, the influence of Bourbaki in choice of material is clear enough, but the insistence on old-fashioned explicit computational proofs at almost every step, proofs only valid for finite dimensional spaces, is hard to justify. The basic result on lifting bilinear mappings to linear ones is proved using bases, but then the existence of $A\otimes B$ is omitted. Eventually, tensors (their duals) show up as multilinear functionals. Characteristic zero is required for exterior algebra, although saying "alternating" instead of "antisymmetric" easily eliminates difficulties with characteristic two. The various isomorphisms between spaces of skew tensors are worked out in great detail using exterior and interior products. Topics omitted are the canonical form for 2-forms, the quadratic relations for Plücker coordinates, any hint of the applications in differential geometry, homological algebra, topology, etc.

Chapters 6 and 7 on real euclidean spaces and the spectral theorem for self-adjoint transformations are very clear and geometrical. Again the restriction to finite dimensions leaves out many interesting examples. The Bessel inequality is not mentioned, but there is a section on lowering and raising indices of tensors in the presence of a metric tensor.

In Ch. 8 (Symmetric bilinear forms), a "quadratic form" on a finite dimensional real space is a continuous functional Φ satisfying $\Phi(x+y)+\Phi(x-y)=2\Phi(x)+2\Phi(y)$. This is novel and it takes some doing to go from this to the associated symmetric matrix. The index of inertia is handled very neatly. The simultaneous reduction of a pair of quadratic forms having no common root to diagonal form is given a nice proof accredited to Milnor. The author's preliminary reduction to the case where one of the forms is non-singular is incorrect (§ 8.10), but the device of taking linear combinations fixes it.

Ch. 9 (Second degree surfaces) is concerned with the classification of real quadratic forms under the affine and euclidean groups. In Ch. 10, the author does the material of Chapters 6 and 7 for the complex case. The treatment is brief and ends with the spectral theorem for normal operators.

The final Chapter 11 (Invariant subspaces) covers the theory of similarity. It is more algebraic than any other in the book. The decomposition of a linear space relative to a given transformation into cyclic subspaces is fairly involved, as is usual. The Cayley-Hamilton theorem finally comes as a consequence of the rational canonical form.

The format and printing are excellent, as is invariably the case in the "Grundlehren" series. There are very few errors. We only mention the absence of the factor 2 in the left hand side of the equation in Problem 3, p. 64.

H. Flanders (Berkeley, Calif.)

3884:

✱Gantmacher, F. R. Matrizenrechnung. I. Allgemeine Theorie. Hochschulbücher für Mathematik. Bd. 36. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958. xi+324 pp. DM 26.80.

Translation of the first part (the first ten chapters) of "The theory of matrices" [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; MR 16, 438].

3885:

Holladay, John C.; and Varga, Richard S. On powers of non-negative matrices. Proc. Amer. Math. Soc. 9 (1958), 631–634.

Let A be an n by n matrix consisting of non-negative elements and such that some (finite) power of A has only positive elements. Let $\gamma(A)$ be the smallest positive integer h such that A^h has only positive elements. The authors prove

$$\gamma(A)\leqq n^2-n+2,$$

a result stated but not proved by Wielandt. They also prove that if $d>0$ of the diagonal elements of A are positive, then

$$\gamma(A)\leqq 2n-d-1,$$

another result of Wielandt. They show that these inequalities may be improved when there are many off-diagonal, non-zero elements.

B. W. Jones (Boulder, Colo.)

3886:

Vakselj, Anton. Algebraische Grundlage der Vektorrechnung. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske Ser. II. 12 (1957), 161–169. (Serbo-Croatian summary)

The matrices considered below are of type $n\times n$; E denotes the unit matrix; and A_1 denotes the trace of A. For any given numbers a_1, a_2, b_1, b_2, c, d, the author defines the 'outer product' $A\pi B$ and the 'inner product' ApB of the matrices A and B by the equations

$$(*)\quad A\pi B=a_1AB+a_2BA+b_1A_1B+b_2B_1A+\{2c(AB)_1+dA_1B_1\}E,$$

$$ApB=2c(AB)_1+dA_1B_1.$$

He then deduces a few properties of these products, all of which follow readily from the definitions. Furthermore, if, on the right-hand side of (*), AB, BA, B, A, E are replaced by $A\pi'B$, $B\pi'A$, $\frac{1}{2}(B\pi'E+E\pi'B)$, $\frac{1}{2}(A\pi'E+E\pi'A)$, $E\pi'E$, respectively, then the resulting expression is denoted by $A\pi\pi'B$. The author shows that the multiplication of π's so defined possesses a faithful representation in terms of certain 6×6 matrices. He concludes the paper with some remarks on the representation of three-dimensional vectors by matrix algebras.

L. Mirsky (Sheffield)

3887:

Huzurbazar, M. S. Eigenvalues and canonical forms of matrices with real quaternion elements. Math. Student 25 (1957), 129–142.

This paper seems to the reviewer to be a compilation of known elementary results on matrices of quaternions. Some of the theorems remain valid if the field is any field with a proper anti-automorphism. To the author's list of references should be added N. Jacobson, Theory of rings, Amer. Math. Soc., New York, 1943 [MR 5, 31].

J. L. Brenner (Menlo Park, Calif.)

3888:

Takeno, Hyôtirô. Contributions to the theory of sedenions. I, II, III. Tensor (N.S.) 7 (1957), 143–172; 8 (1958), 21–37.

Let γ_α ($\alpha=1, \cdots, 5$) be Dirac matrices. Every matrix S (all matrices are to be 4×4) has a unique representation as a linear combination of the base matrices $\gamma_{\alpha\beta}\equiv\gamma_{[\alpha}\gamma_{\beta]}$, γ_α, I:

$$S=2^{-1}s^{\alpha\beta}\gamma_{\alpha\beta}+s^\alpha\gamma_\alpha+sI$$
$$=2^{-1}s^{ab}\gamma_{ab}+s^5\gamma_5+sI+s^a\gamma_a+s^{a5}\gamma_{a5}$$

($\alpha, \beta=1, \cdots, 5$; $s^{\alpha\beta}=-s^{\beta\alpha}$; $a, b=1, \cdots, 4$; $\gamma_{a5}=\gamma_a\gamma_5$). ($s^{\alpha\beta}$, s^α, s) are the coefficients of expansion for S. The author investigates in great detail the relation of general and special matrices to their coefficients of expansion and to related quantities, among them the following: g_{ij} ($i, j=1, \cdots, 4$), where $g_{ij}dx^idx^j$ is the line element of some 4-dimensional space; $s_{ij}^*=2^{-1}\,\varepsilon_{ijkl}s^{kl}$, where ε_{ijkl} is the tensor antisymmetric in every pair of indices, with ε_{1234} given; $p=s_{ij}s^{ij}$, $q=s_{ij}s^{*ij}$. Most of the results involve notation too lengthy to reproduce here, so this review will be restricted (for the most part) to general statements.

Chapter 1 (part I) gives numerous formulas for $M^{ij}\gamma_{ij}$, $M^{ijk}\gamma_{ijk}$, etc. (M^{ij}, M^{ijk} arbitrary tensors), and for antisymmetric tensors s_{ij}, s_{ij}^*; conditions for commutativity or anticommutativity of matrices; conditions for the holding of equations of the form $(\mu^{ij}-\mu g^{ij})a_j=0$, where s_{ij}, a_i, μ are, respectively, an antisymmetric tensor, a vector, and a scalar, and $\mu^{ij}=s^{ij}-\eta s^{*ij}$.

In Chapter 2, A-type matrices are investigated: A is A-type if $A=2^{-1}s^{ij}\gamma_{ij}+s^5\gamma_5+sI$ ($s^{ij}=-s^{ji}$). An A-type matrix can be expressed as $\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}$ where A_1, A_2 are non-zero 2×2 matrices. If $|A|\neq 0$ then A^{-1} is also A-type and is given by $A^{-1}=2^{-1}(-\alpha_1s^{ij}+\alpha_2s^{*ij})\gamma_{ij}+(\alpha_1s_5+\alpha_2)\gamma_5+(\alpha_1s+\alpha_2s_5)I$, with $\alpha_1=2^{-1}(|A_1|^{-1}+|A_2|^{-1})$, $\alpha_2=2^{-1}(|A_1|^{-1}-|A_2|^{-1})$. An A-type matrix is called $\mathfrak{A}$-type [$\overline{\mathfrak{A}}$-type] if $\alpha_2=0$[$\alpha_1=0$]. Such matrices enter into theorems like the following: $A^{-1}\gamma^iA$ or $A\gamma^iA^{-1}$ is a linear combination of γ^i [of γ^{i5}] if and only if A is of $\mathfrak{A}$-type [$\overline{\mathfrak{A}}$-type]. Chapter 3 deals similarly with B-type matrices: B is B-type if $B=s^i\gamma_i+s^{i5}\gamma_{i5}$, and can then be written as $\begin{bmatrix}0 & B_1\\ B_2 & 0\end{bmatrix}$. If $|B|\neq 0$, then B^{-1} is also B-type and is given by $(\beta_1s^i-\beta_2s^{i5})\gamma_i+(\beta_2s^i-\beta_1s^{i5})\gamma_{i5}$, with $\beta_1=2^{-1}(|B_1|^{-1}+|B_2|^{-1})$, $\beta_2=2^{-1}(|B_1|^{-1}-|B_2|^{-1})$; and B is called $\mathfrak{B}$-type [$\overline{\mathfrak{B}}$-type] if $\beta_2=0$ [$\beta_1=0$]. $B^{-1}\gamma^iB$ [$B\gamma^iB^{-1}$] is a linear combination of γ^i [of γ^{i5}] if and only if B is $\mathfrak{B}$-type [$\overline{\mathfrak{B}}$-type].

Chapter 4 (part II) deals with the general matrix S. Its determinant and inverse are expressed in terms of the coefficients of expansion of S, as are also the matrices $S^{-1}\gamma_5S$, $S^{-1}\gamma_iS$, etc. Necessary and sufficient conditions are found that S be (resp.) A-type, B-type, $\mathfrak{A}$-type, etc. Finally, the nature of solutions S to each of the following equations is determined: $S\gamma_i=\eta\gamma_{i5}S$, $S\gamma_i=\eta S\gamma_{i5}$, $\cdots$, $S^2=0$, $S^2=\alpha I$, $S^2=I$, $S^2=\gamma_5$. In particular, the cases where a solution S is symmetric or antisymmetric are discussed.

Let γ_i be a linear combination of γ_α and let $|S|\neq 0$. Set $\gamma_i'=S^{-1}\gamma_iS$. If γ_i' is also a linear combination of γ_α then S is $\mathfrak{A}$-type or $\mathfrak{B}$-type. Setting $\gamma_i'=P_i^{\cdot j}\gamma_j$, then (*) $g_{lm}P_i^{\cdot l}P_j^{\cdot m}=g_{ij}$, $|P_i^{\cdot j}|=\pm 1$, and there is a 1-1 correspondence between P_{ij} and matrices of $\mathfrak{A}$-type or of $\mathfrak{B}$-type. In Part III these correspondences are studied in detail, according to various cases. A typical result of this kind is this: There is a 1-1 correspondence between all P_{ij} satisfying $P_{ij}=-P_{ji}$ and all $\mathfrak{A}$-type matrices for which $s=\eta s_5$, $s_{ij}=\eta s_{ij}^*$.

I. M. Sheffer (University Park, Pa.)

3889:

Howarth, J. C. On the real rotation group. Quart. J. Math. Oxford Ser. (2) 7 (1956), 241–243.

This paper is concerned with the Euler-Hurwitz parametrisation of the rotation group [A. Hurwitz, Mathematische Werke, II, Basel, Birkhäuser, 1933; p. 551], i.e., the factorization of an n-dimensional rotation R into product of $\frac{1}{2}n(n-1)$ plane rotations. With suitable range restrictions, the angles of these plane rotations are uniquely determined by R. Schur [S.-B. Preuss. Akad. Wiss. Berlin. Phys.-Math. Kl. (1924), 189–208; p. 196] found an error in Hurwitz's angle restrictions, and gave modified conditions to secure uniqueness. The angle restrictions given in this paper are slightly different from Schur's. According to the author, there was a minor error in Schur's conditions.

Ky Fan (Notre Dame, Indiana)

3890:

Khan, N. A. The characteristic roots of the product of two matrices. Tôhoku Math. J. (2) 9 (1957), 234–237.

The paper is concerned with commuting pairs of matrices with complex elements, but does not make use of Frobenius' theorem on the eigenvalues of their product. Upper bounds for the real and imaginary parts of the eigenvalues of the product are determined.

O. Taussky-Todd (Pasadena, Calif.)

3891:

Khan, N. A. The characteristic roots of the product of matrices. Quart. J. Math. Oxford Ser. (2) 7 (1956), 138–143.

It is known that a bound for the absolute values of the characteristic roots of a finite matrix of complex numbers is given by the largest row [column] sum, i.e., the sum of the absolute values of the elements in a row [column]. Here, two expressions in terms of the row and column sums of two matrices are given which majorize the eigenvalues of their product.

O. Taussky-Todd (Pasadena, Calif.)

3892:

Marathe, C. R. A note on characteristic values of products of two matrices. Quart. J. Math. Oxford Ser. (2) 8 (1957), 291–294.

Bounds for eigenvalues and singular values of products of rectangular matrices are considered [cf. #3891, reviewed above]. *O. Taussky-Todd* (Pasadena, Calif.)

3893:

Gun, G. Limits for the characteristic roots of a matrix. I. Advancement in Math. 4 (1958), 450–456. (Chinese)

Let $A=(a_{ij})$ be a real or complex n-square matrix. Let $R_i=\sum_j |a_{ij}|$, $C_i=\sum_j |a_{ji}|$. Let S and T denote the largest and the least number among the R_i's. Let $L=\max_i R_iC_i$, $M=n^{-1}\sum_i R_iC_i$ and $N=\min_{i\neq j}|a_{ij}|$. The main result asserts that every eigenvalue λ of A satisfies the inequality

$$|\lambda|^2 \leqq L-\frac{T}{S}\cdot\frac{(L-M)N^2}{(S-T+N)^2},$$

which improves the known inequality $|\lambda|^2\leqq L$ of E. W. Barankin [Bull. Amer. Math. Soc. 51 (1945), 767–770; MR 7, 107]. Several other related upper bounds for $|\lambda|$, $|\mathrm{Re}\ \lambda|$, $|\mathrm{Im}\ \lambda|$ are obtained. There are also inequalities concerning the maximal eigenvalue of a matrix with non-negative elements. *Ky Fan* (Notre Dame, Ind.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 3797, 3801.

3894:

Nagahara, Takasi; and Tominaga, Hisao. On Galois theory of division rings. II. Math. J. Okayama Univ. 7 (1957), 169–172.

Let K be a division ring and L a division subring of K. K/L is said to be locally Galois if for any finite subset F of K there exists a division subring L' containing $L(F)$ that is Galois and finite over L. The main theorem of the paper states: Let K/L be Galois and locally Galois, and $[K:V_K(V_K(L))]_l\leqq\aleph_0$ (V_K: centralizer in K). If K' is an intermediate division ring of K/L with $[V_K(L):V_K(K')]<\infty$, then K/K' is Galois and locally Galois. Every ring L-isomorphism of K onto a similar intermediate ring may be induced by an element of the Galois group of K/L.

T. Nakayama (Nagoya)

3895:

Kertész, A. Über die allgemeine Theorie linearer Gleichungssysteme. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) 1(49) (1957), 303–307.

The paper represents a lecture delivered at the Romanian Math. Congress, Bucharest, 1956. Proofs of the results are to be published separately. The following extracts may serve to indicate the character of the new results, and their proper place with reference to the established theory.

"Es sei R ein beliebiger (assoziativer) Ring und G ein mit R als linksseitigem Operatorbereich versehener R-Modul. . . . Unsere weiteren Ergebnisse über algebraisch abgeschlossene Moduln fassen wir in dem folgenden Satz zusammen: α) Ein beliebiger R-Modul $\mathfrak{G}$ besitzt eine algebraisch abgeschlossene Erweiterung; β) Für einen R-Modul $\mathfrak{G}$ sind die folgenden Aussagen äquivalent: β_1) $\mathfrak{G}^*$ ist eine maximale algebraische Erweiterung von $\mathfrak{G}$; β_2) $\mathfrak{G}^*$ ist eine algebraisch abgeschlossene algebraische Erweiterung von $\mathfrak{G}$; β_3) $\mathfrak{G}^*$ ist eine minimale algebraisch abgeschlossene Erweiterung von $\mathfrak{G}$; γ) Jeder R-Modul $\mathfrak{G}$ besitzt eine und bis auf einen Isomorphismus nur eine Erweiterung $\mathfrak{G}^*$, welche die Eigenschaften β_1), β_2), β_3) besitzt, ...

"Der Begriff des algebraisch abgeschlossenen Moduls fällt im Falle unitärer Moduln zusammen mit dem Begriff des kompletten Moduls von R. Baer, und mit dem Begriff des injektiven Moduls. (Ein R-Modul $\mathfrak{G}$ ist unitär falls R ein Einselement besitzt, das auf die Elemente von $\mathfrak{G}$ identisch operiert) . . . .

"Somit sind in den obigen Sätzen die Hauptergebnisse folgender früheren Arbeiten als Spezialfälle enthalten und im Rahmen einer umfassenderen Theorie untergebracht: R. Baer, Bull. Amer. Math. Soc. 46 (1940), 800–806 [MR 2, 126]; B. Eckmann und A.Schopf, Arch. Math. 4 (1953), 75–78 [MR 15, 5]; S. Gacsályi, Publ. Math. Debrecen 2 (1952), 292–296 [MR 15, 775]."

A. J. Kempner (Boulder, Colo.)

3896:

Dieudonné, Jean. Remarks on quasi-Frobenius rings. Illinois J. Math. 2 (1958), 346–354.

Let A be a ring and E a left A-module. E is said to have perfect duality if the following conditions hold for its dual (in Bourbaki's sense) E^* and the orthogonal M^0 of its submodules M: (A) $E\to E^{**}$ is an isomorphism; (B) $E^*/M^0\to M^*$ is an isomorphism; (C) $M^{00}=M$; (D) $(M\cap N)^0=M^0+N^0$. After some preliminary considerations on mutual implications among these conditions, the paper shows that, for an artinian ring A, any two of the following three conditions imply the third and imply that finitely generated A-modules have perfect duality: (m_s) the dual of any simple left A-module has length $\leqq 1$; (m_d) similarly with right modules; (e) the left A-module A and right A-module have the same length. [A somewhat weaker result has independently been obtained by Morita and Tachikawa, Math. Z. 65 (1956), 414–428; MR 20 #1704]. It follows that all finitely generated modules over a noetherian ring A have perfect duality if and only if A is a quasi-Frobenius ring. A counterpart of the result for Frobenius rings is obtained on replacing lengths with "colengths" and on using a result of Ikeda [Osaka Math. J. 3 (1951), 227–239; MR 13, 719]. It is proved further that an algebra of finite rank satisfying the condition (m_s) is quasi-Frobenius, using a result of the reviewer [Proc. Japan Acad. 25 (1950), no. 7, 45–50; MR 12, 797]. *T. Nakayama* (Nagoya)

3897:

Ghika, Al. Modules libres sur des algèbres. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 8 (1956), 509–516. (Romanian. Russian and French summaries)

3898:

Bialynicki-Birula, A. On the spaces of ideals of semirings. Fund. Math. 45 (1958), 247–253.

Theorem (1): Let R be a semiring in which every principal ideal has a central generator, and let $\mathscr{J}$ be a set of prime ideals, endowed with the Stone topology; then $\mathscr{J}$ is a Hausdorff space if and only if every prime ideal containing $\cap\mathscr{J}$ is contained in at most one member of $\mathscr{J}$. The necessity does not require the condition about generators, and hence it generalizes a result of the reviewer for rings [Fund. Math. 45 (1957), 1–16; MR 19, 1156]. The main result in the paper under review is the sufficiency; this is established by making use of McCoy's concept and results about m-systems [Amer. J. Math. 71 (1949), 823–833; MR 11, 311], extended to semirings. Another theorem is (2): If R is as above, then the space $\mathscr{Q}$ of all prime ideals is Hausdorff if and only if the set $\{Q\in\mathscr{Q}: a\in Q\}$ is open for every $a\in R$. This generalizes a result of the reviewer's for commutative rings [loc. cit.]. The author also includes a discussion of topologies on families of sets of finite character; the proof of (2) is based upon part of this discussion.

L. Gillman (Princeton, N.J.)

3899:

Bourne, Samuel; and Zassenhaus, Hans. On the semiradical of a semiring. Proc. Nat. Acad. Sci. U.S.A. 44 (1958), 907–914.

For a semiring S a two-sided ideal $\sigma(S)$ called the semiradical of S is defined. It is an analogue of the (Jacobson) radical of a ring and (properly, in general) contains the Jacobson radical $R(S)$ of S defined formerly [Bourne, same Proc. 37 (1951), 163–170; MR 13, 7]. Thus, by the equivalence relation $\sim$ defined for a, b in S by the existence of an $x\in S$ with $a+x=b+x$, a semiring S^* with cancellation law is obtained. $\sigma(S)$ is the counter-image of the Jacobson radical $R(S^*)$ of S^*. It is also the maximal right ideal I of S in which, for every a, b in I, elements a', b' of S with $a+a'+aa'+bb'=b+b'+ba'+ab'$ can be found. Besides this (and another) characterization from below, two characterizations of $\sigma(S)$ from above are given, namely, as the intersection of all "semimaximal" and "semimodular" left ideals and as the intersection of all "semiprimitive" ideals; the terms in quotation marks are analogues of "maximal", "modular" and "primitive" in ring theory. The difference semiring of a semiring over its semiradical has 0 semiradical. Under certain chain condition $\sigma(S)$ is seminilpotent (i.e., nilpotent modulo the zeroid).

T. Nakayama (Nagoya)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

See also 3906, 3913.

3900:

Koecher, Max. Analysis in reellen Jordan-Algebren. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa 1958, 67–74.

Let A be a Jordan algebra with coefficient field K, $\{u_1, \cdots, u_n\}$ be a basis for A over K, and $u_k\circ u_l=\sum_m h_{klm}u_m$. Then the n-rowed square matrices $L_m=(h_{mlk})$ satisfy $u\circ v=L(u)v=L(v)u$, where $L(u)=\sum_m u^{(m)}L_m$; u is any element with components $u^{(m)}$. If $L(u)$ is symmetric for all u, then A is called a symmetric algebra; A is called weakly-associative if $u\circ(u^{-1}\circ v)=u^{-1}\circ(u\circ v)$, where $u^{-1}=L^{-1}(u)c$ and c is the unit vector; u^{-1} exists if $\det L(u)\neq 0$, or, equivalently, if u is not a zero-divisor.

Henceforth, K is the field R of reals. Theorems: Every weakly-associative algebra of finite rank over R is a Jordan algebra; every symmetric Jordan algebra of finite rank over R is weakly-associative; A symmetric algebra of finite rank over R is weakly-associative if and only if x^{-1} is the local gradient of a scalar function of x. If $A=R^n$ is a symmetric Jordan algebra, set $e(x)=e^{L(x)}c$; the properties of this function are studied. In a subsequent paper the results are to be applied to questions of the uniqueness of geodesics in R^n.

P. M. Whitman (Baltimore, Md.)

3901:

Jacobson, N. A note on three dimensional simple Lie algebras. J. Math. Mech. 7 (1958), 823–831.

Among the simple Lie algebras of dimension 3 over an arbitrary field of characteristic different from 2, the "split" algebra L, i.e., the one with basis e, f, h such that $[ef]=h$, $[eh]=2e$, $[fh]=-2f$, is of special interest. It is the only such Lie algebra over an algebraically closed field and plays an important part in the theory of Lie and Jordan algebras [Jacobson, Proc. Amer. Math. Soc. 2 (1951), 105–113; MR 14, 241; Math Ann. 136 (1958), 375–386]. In the case of characteristic zero, it is well known that every finite-dimensional representation of a semi-simple Lie algebra is completely reducible. In the case of prime characteristic p, it has been shown by the author [Amer. J. Math. 74 (1952), 357–359; MR 13, 816] that every Lie algebra has a representation which is not completely reducible, and Hochschild [Proc. Amer. Math. Soc. 5 (1951) (1954), 603–605; MR 16, 562] has shown that the same is true, except for certain commutative algebras, when one considers only "restricted" representations of restricted Lie algebras. The author here shows that for the algebra L above, every representation in which e and f are represented by nilpotent transformations of index less than the characteristic p of the base field is completely reducible, and that this seems to be the best possible result of this type. The structure of the most general associative algebra with identity, and with generators e, f, h satisfying $eh-he=2e$, $fh-hf=-2f$, $ef-fe=h$, $f^m=0=e^m$, where $m\leq p-1$ if $p\neq 0$, is determined; it is a direct sum of m full matrix algebras, namely the j by j matrices over the base field for each $j\leq m$. An application is made to general "split" Lie algebras {it seems that the property (4), which is used explicitly in the proof of this theorem (theorem 3) should be added to the hypotheses}. A classification of simple 3-dimensional algebras in terms of equivalence classes of 3 by 3 non-singular symmetric matrices under "multiplicative cogredience" is also given.

G. B. Seligman (Münster)

3902:

Kokoris, Louis A. On nilstable algebras. Proc. Amer. Math. Soc. 9 (1958), 697–701.

This paper is another contribution to the study of simple commutative power-associative algebras of degree two [Trans. Amer. Math. Soc. 74 (1953), 323–343; Ann. of Math. (2) 64 (1956), 544–550; MR 14, 614; 18, 375]. Let A be such an algebra over F of characteristic $\neq 2, 3$, and $1=u+v$ for orthogonal idempotents u, v. Write $A=A_1+A_{12}+A_2$ with $A_1=A_u(1)$, $A_{12}=A_u(\frac{1}{2})$, $A_2=A_u(0)$, where $A_u(\lambda)=\{x|xu=\lambda x\}$. Then $A_1=uF+G_1$, $A_2=vF+G_2$, where each G_i is a nilalgebra. The author calls u a nilstable idempotent, and A nilstable with respect to u in case $A_{12}A_i\subseteq A_{12}+G_{3-i}$ for $i=1, 2$. Theorem: A is a Jordan algebra if and only if A is nilstable

with respect to two idempotents u, f such that $u\neq 1$, $f\neq 1$, $u+f\neq 1$ and such that f is not of the form $f=u+w_{12}+w_1+w_2$ or $f=v+w_{12}+w_1+w_2$, with w_{12} in A_{12}, w_1 in G_1, w_2 in G_2. *R. D. Schafer* (Princeton, N.J.)

3903:

Sce, Michele. Sulla varietà dei divisori dello zero nelle algebre. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 23 (1957), 39–44.

Let A be an alternative algebra over a field of characteristic different from 2 and let $V(A)$ be the linear variety of zero divisors of A. If A is simple, let M be the set of primitive algebras which are direct factors of A and let D be an element of M of maximal order d. Then the author proves that dim $V(A)=n-d$, where n is the order of A. This result is used to obtain the main theorem of the note: For any A, not necessarily simple, dim $V(A)=n-q$, where n is the order of A and q is the order of the smallest primitive factor of the simple components of A. *R. L. San Soucie* (Buffalo, N.Y.)

3904:

van Albada, P. J. Two theorems about quadratic nonassociative algebras. Nederl. Akad. Wetensch. Proc. Ser. A 61=Indag. Math. 20 (1958), 319–321.

A quadratic algebra is called symmetric if there is an involutary skew automorphism $T(a)=\bar{a}$ in A which leaves the elements of F, and only these, invariant. It is proved that a flexible quadratic algebra of characteristic not two is symmetric. Also an example is given to show that there exist nonflexible symmetric division algebras of dimension four. *L. A. Kokoris* (Chicago, Ill.)

3905:

Luchian, T. Algèbres par rapport au corps des nombres réels d'ordre 2, sans diviseurs de zéro. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) 3 (1957), 19–30. (Romanian. Russian and French summaries)

The author proves that an algebra (not assumed associative) as described by the title and having an identity element, must be the field of complex numbers. *B. Harris* (Evanston, Ill.)

HOMOLOGICAL ALGEBRA

See also 4263.

3906:

Leger, George F., Jr. On cohomology of Lie algebras. Proc. Amer. Math. Soc. 8 (1957), 1010–1020.

For any Lie algebra G, ideal L of G, and G-module M, the cohomology-group $H^n(L, M)$ may be defined as a G-module in a natural way [cf. G. Hochschild and J.-P. Serre, Ann. of Math. (2) 57 (1953), 591–603; MR 14, 943]. The author examines this definition for the usual interpretations of H^n for low n. More precisely, he defines G-module structures on $\mathrm{Ext}_L(P, Q)$ (where P and Q are G-modules), $\mathrm{Ext}(M, L)$ and $\mathrm{Kern}(L, M)$, the space of similarity classes of L-kernels with nucleus M [cf. Hochschild, Amer. J. Math. 76 (1954), 698–716; MR 16, 109], which make the natural mappings into the cohomology groups into G-mappings (G-isomorphisms for $n=1, 2$). An appendix sketches the corresponding result for groups. *P. M. Cohn* (Manchester)

3907:

Sridharan, R. On some algebras of infinite cohomological dimension. J. Indian Math. Soc. (N.S.) 21 (1957), 179–183.

The algebra Γ of power series in n variables over a field has infinite homological dimension, because if Λ is the field of quotients of Γ, then dim $\Gamma\geqq$dim $\Lambda\geqq$transcendence degree of Λ, which is infinite. *D. Zelinsky* (Evanston, Ill.)

GROUPS AND GENERALIZATIONS

See also 3799a–b, 3800, 3853, 3889, 3930, 4220, 4231, 4260.

3908:

Moran, S. Associative operations on groups. I. Proc. London Math. Soc. (3) 6 (1956), 581–596.

Utilizing verbal subgroups [see B. H. Neumann, Math. Ann. 114 (1937), 506–525; or the following review], the author constructs a set of new products of groups called verbal products. They are, in a sense, intermediate between the direct and free product. Kuroš originally posed the problem of finding products of groups, in addition to the free and direct product, which (among other things) are associative and commutative. Such products might be expected to be useful in classifying known groups and constructing new groups (cf. the importance of the direct product in finite Abelian groups).

Let G_α, $\alpha\in M$, be an arbitrary set of groups; F, the free product of the G_α; $V(F)$, a specified verbal subgroup [Neumann, loc. cit.] of F; $[G_\alpha]^F$, the normal subgroup generated by $g_\alpha g_\beta g_\alpha^{-1}g_\beta^{-1}=[g_\alpha, g_\beta]$, $g_\alpha\in G_\alpha$, $g_\beta\in G_\beta$, $\alpha\neq\beta$, $\alpha, \beta\in M$. The verbal product of the G_α is $F/V(F)\cap[G_\alpha]^F$. For example, the verbal products associated with the words $[x, y]$, xx^{-1}, x are the direct product, the free product, and the direct product, respectively. The nilpotent products of Golovin [Mat. Sb. N.S. 27(69) (1950), 427–454; MR 12, 672] and the products of the reviewer [Trans. Amer. Math. Soc. 81 (1956), 425–452; MR 17, 105] are special cases of verbal products. The author proves the associativity of verbal products and other properties: for example, if every G_α satisfies a specified identical relation, so does the corresponding verbal product of the G_α, and this verbal product is a maximal group satisfying this relation (in a sense specified by the author).

A regular product [according to Golovin, loc. cit.] of the G_α, $\alpha\in M$, is of the form F/N, where $N\subseteq[G_\alpha]^F$. All verbal products are regular products, but not vice-versa. The author constructs a commutative, non-associative regular product, thus answering a question raised by Golovin. This product is different from the commutative, non-associative products of the reviewer [loc. cit.]. Golovin also asked if $[G_\alpha]^F$ is a free group, which the author answers affirmatively by referring to the Kuroš subgroup theorem, and the rank of N is given. [See also Benado, Math. Nachr. 14 (1955), 213–234; MR 18, 871.] The author also proves some additional theorems about verbal products and verbal subgroups.

[Other work on regular products can be found in Fridman, Dokl. Akad. Nauk SSSR 109 (1956), 710–712; MR 18, 279; Benado, C. R. Acad. Sci. Paris 243 (1956), 1092–1093; MR 18, 279.] *R. R. Struik* (Vancouver, B.C.)

3909:

Moran, S. Duals of a verbal subgroup. J. London Math. Soc. 33 (1958), 220–236; corrigendum, 34 (1959), 250.

A verbal subgroup $V_f(G)$ [Neumann, Math. Ann. 114 (1937), 506–525] of a group, G, is the subgroup generated by $f(x_1, x_2, \cdots, x_n)$, $x_i \in G$, where $f(x_1, \cdots, x_n)$ is a fixed word in the indeterminates x_i. The corresponding marginal subgroup, $M_f(G)$ [Hall, J. Reine Angew. Math. 182 (1940). 156–157; MR 2, 125], is the set of all $\xi \in G$ satisfying all relations of the form $f(x_1\xi, x_2, \cdots, x_n) = f(x_1, x_2\xi, x_3, \cdots, x_n) = \cdots = f(x_1, \cdots, x_n\xi) = f(x_1, x_2, \cdots, x_n)$. Let $Z_f(G)$ be the centralizer of $V_f(G)$. The author lists a large number of properties of these subgroups indicating a dual relationship between $V_f(G)$ on one hand, and $M_f(G)$ and $Z_f(G)$ on the other: the union of verbal subgroups is a verbal subgroup, while the intersection of marginal subgroups [centralizers of verbal subgroups (CVSs)] is a marginal subgroup [centralizer of a verbal subgroup (CVS)]. Neumann [loc. cit.] gave an example showing that the intersection of verbal subgroups need not be a verbal subgroup. The author gives a simpler example and also constructs examples showing that the union of marginal subgroups [CVSs] need not be a marginal subgroup [CVS]. A result stated by Hall [loc. cit.], relating $M_{[f,g]}(G)$ to $M_f(G)$ and $M_g(G)$, where $[f, g] = f^{-1}g^{-1}fg$, is proved, as well as a corresponding theorem for CVSs and the dual theorem for verbal subgroups. It is proved that a non-Abelian normal subgroup of a free product has a trivial centralizer, and this is used to show that if G is a free product (except for the infinite dihedral group), then $M_f(G) = Z_f(G) = 1$, while $V_f(G) = 1$ if and only if G is the infinite dihedral group.

The author generalizes these two duals of verbal subgroups: let $h(u, v)$ be a word in u and v, then $H_f^1(G)$ is the least subgroup containing all $\xi \in G$ such that $f(h(x, \xi), x_2, \cdots, x_n) = \cdots = f(x_1, \cdots, x_{n-1}, h(x_n, \xi)) = f(x_1, \cdots, x_n)$ for all $x_i \in G$; $H_f^2(G)$ is the least subgroup containing all $z \in G$ such that $h(f(x_1, \cdots, x_n), z) = 1$ for all $x_i \in G$. If $h(u, v) = uv$, then $H_f^1(G)$ becomes $M_f(G)$; if $h(u, v) = [u, v]$, $H_f^2(G)$ becomes $Z_f(G)$. $K_f^1(G)$ and $K_f^2(G)$ are defined by substituting "some set of $x_i \in G$" for "all $x_i \in G$". If $h(u, v) = v^{-1}uv$, $H_f^1(G)$ is called the invariable subgroup $I_f(G)$. $H_f^1(G)$, including $I_f(G)$, has most of the properties of $M_f(G)$ and $Z_f(G)$, and $1 \leqq M_f(G) \leqq I_f(G) \leqq Z_f(G) \leqq G$ for all f and G. If $g(x_1, \cdots, x_n, x) = [f(x_1, \cdots, x_n), x]$, then $Z_f(G) \leqq Z_g(G)$, $I_f(G) \leqq I_g(G)$ and $M_f(G) \leqq M_g(G)$; other relationships between these subgroups are investigated. Other possibilities for $h(u, v)$ are mentioned.

R. R. Struik (Vancouver, B.C.)

3910:

Moran, S. The homomorphic image of the intersection of a verbal subgroup and the cartesian subgroup of a free product. J. London Math. Soc. 33 (1958), 237–245; corrigendum, 34 (1959), 250.

If $F = \prod^* G_\alpha$, $\alpha \in M$, is the free product of the groups G_α, then the Cartesian subgroup $[G_\alpha]^F$ of F is the normal closure of the subgroup generated by all $[G_\alpha, G_\beta] = \{g_\alpha^{-1}g_\beta^{-1}g_\alpha g_\beta, g_\alpha \in G, g_\beta \in G, \alpha \neq \beta, \alpha, \beta \in M\}$. Let $V(F)$ be a verbal subgroup of F. The author studies $C_V[G_\alpha]^F = V(F) \cap [G_\alpha]^F$.

In a previous paper [#3908 above] the author proved that if ϕ is a homomorphism of F whose kernel is contained in $[G_\alpha]^F$, then $C_V[G_\alpha]^F\phi = C_V[G_\alpha\phi]^{F\phi}$. Here the author gives an example to show that the equality is not valid for an arbitrary homomorphism: if F is the free group of rank 2 with a and b as generators, and ϕ is the homomorphism of F onto $G = \{a, b, a^3 = [a, b]\}$, then $(F^3 \cap [F, F])\phi \neq G^3 \cap [G, G]$, where F^3, G^3 are the verbal subgroups associated with the word x^3.

If x_1 is a commutator word of weight 1 and commutators of weight less than n have been defined, then $f(x_1, \cdots, x_n)$ is a commutator of weight n if $f(x_1, \cdots, x_n) = [f_1(x_1, \cdots, x_i), f_2(x_{i+1}, \cdots, x_n)]$, where f_i are commutator words whose weights are less than n. Let $f(F)$ be the verbal subgroup associated with f. Let G_α^F be the normal closure of G_α in F. The author proves that $f(F) \cap [G_\alpha]^F = f[G_\alpha^F] = f[G_\alpha]^F$, where $f[G_\alpha^F]$ is the subgroup generated by all possible subgroups $f(G_{\alpha_1}^F, \cdots, G_{\alpha_n}^F)$, $\alpha_i \in M$, with the exception of the $f(G_\alpha^F) = f(G_\alpha^F, \cdots, G_\alpha^F)$. $f[G_\alpha]^F$ is defined inductively on the weight of n: for $n = 1$ or 2, $f[G_\alpha]^F = [G_\alpha]^F$; otherwise,
$f[G_\alpha]^F =$

$$\prod_{\alpha \neq \beta} [f_1(G_\alpha), f_2(G_\beta)]^F \cdot [f_1(F), f_2[G_\alpha]^F] \cdot [f_1[G_\alpha]^F, f_2(F)].$$

The author also states two unsolved problems related to this subject.

R. R. Struik (Vancouver, B.C.)

3911:

MacLane, Saunders. A proof of the subgroup theorem for free products. Mathematika 5 (1958), 13–19.

For each index β in a set J let A_β be a group. The free product G of the A_β is written $G = *_\beta A_\beta$, $\beta \in J$. This paper gives a sharpened version of the Kurosh theorem that a subgroup H of G is itself a free product of a free group F and conjugates of subgroups of the A_β.

The proof depends on a multiple selection of representatives for left cosets of H, subject to certain conditions. For each $\beta \in J$, a representative $R_\beta(C)$ of a left coset C of H is chosen so that $R_\beta(H) = 1$ and $R_\beta(Ca) \in R_\beta(C)A_\beta$ for every $a \in A_\beta$. For a H-A_β double coset D choose a representative $s = S_\beta(D)$, $D = HsA_\beta$. Then for a coset $C \subset D$, choose $R_\beta(C) \in sA_\beta$. Choices for different β's may be related to produce what is called a uniform Schreier system of representatives. This is done by imposing the following condition: Let $g = sa$, $c_1c_2 \cdots c_t a$ being the reduced form of g, $a \in A_\beta$, $c_t \in A_\gamma$; then $R_\beta(C) = g$ implies that s is both the β-representative and the γ-representative of its coset, i.e., $R_\beta(Hs) = s = R_\gamma(Hs)$.

With this machinery, the Kurosh theorem is proved in the following form: $H = F*[*_\beta *_D(H \cap sA_\beta s^{-1}))]$, where $s = S_\beta(D)$, the inner product is the free product of conjugates of subgroups, and F is freely generated by the distinct elements $R_\beta(C)R_\gamma(C)^{-1}$ which are not the identity, β being fixed and for all choices for γ and C. Furthermore, if the index $[G:H]$ is finite, then the number of generators of F is $1 - [G:H] + \sum_\beta ([G:H] - m_\beta)$, where m_β is the number of H-A_β double cosets in G.

Marshall Hall, Jr. (Columbus, Ohio)

3912:

Burton, R. C.; and Connor, W. S. On the identity relationship for fractional replicates of the 2^n series. Ann. Math. Statist. 28 (1957), 762–767.

Consider an Abelian group of order 2^n generated by n letters $A, B, C, \cdots$, each of period 2. A subgroup of order 2^r will comprise r independent elements and all products formed from them. The authors first consider the problem of whether such a subgroup of order 2^r exists when the number of letters, w, in each particular group element is specified. Let X be any one of the $2^r - 1$ elements of the subgroup, other than I; then the subgroup exists, for specified w's, if and only if the system of $2^r - 1$ equations (X ranging), $\sum_O w - \sum_E w = 2^{r-1}t(X)$, can be solved in non-negative integers $t(X)$, and the solution is unique when the t's are given. Here $\sum_O$ and $\sum_E$ denote sums over those

group elements having, respectively, odd and even numbers of letters in common with X; the t's are interpretable as the number of letters in the basic disjoint sets when the r subgroup generators are represented on a Venn diagram.

In a more general case, the authors then suppose that the numbers of letters in group elements are known, but without reference to particular group elements; a necessary condition for the existence of the subgroup is that there exist 2^r-1 non-negative integers t such that $\sum t=n$ and $\sum t^2=2^{2-r}\sum w^2-n^2$. This condition, together with known elementary necessary conditions, is not sufficient. *R. G. Stanton* (Waterloo, Ont.)

3913:

Block, Richard. On torsion-free abelian groups and Lie algebras. Proc. Amer. Math. Soc. 9 (1958), 613–620.

Following the method of Albert and Frank [Univ. e Politec., Torino. Rend. Sem. Mat. 14 (1954–1955), 117–139; MR 18, 52], the author defines, for every abelian group G, an algebra $L=L(G, g, f)$ over a field F by using an additive mapping $g:G\rightarrow F$ and an alternate biadditive mapping: $G\times G\rightarrow F$, as follows: L has a basis $\{u_\alpha|\alpha\in G\}$ and the multiplication table $u_\alpha u_\beta=(f(\alpha, \beta)+g(\alpha-\beta))u_{\alpha+\beta}$. Necessary and sufficient conditions on f and g for $L(G,g,f)$ to be a simple Lie algebra are given. This implies, in particular, that if L is a simple Lie algebra of characteristic zero, then G is torsion-free. And for any torsion-free abelian group G there exists a simple Lie algebra $L(G, g, f)$ of characteristic zero, if F has sufficiently large degree over the rationals. The derivations and locally algebraic derivations of L are determined. Using these, it is shown that the simple Lie algebra $L(G, g, f)$ of characteristic zero determines the group G up to isomorphism and determines g and f up to a scalar multiple. No application of $L(G, g, f)$ to the study of the group G is made. *R. Ree* (Vancouver, B.C.)

3914:

Vilyacer, V. G. On the theory of locally nilpotent groups. Uspehi Mat. Nauk (N.S.) 13 (1958), no. 2(80), 163–168. (Russian)

A. I. Mal'cev has defined a group to be of finite rank if there exists a natural number r such that every finitely generated subgroup is contained in a subgroup generated by r elements. Theorem 1: if every abelian subgroup of a group G is of finite rank, and if every subset of two elements is contained in a nilpotent subgroup, then G is locally nilpotent, i.e., every finite subset is contained in a nilpotent subgroup. Theorem 2: if A is an abelian normal subgroup of a group G, if A has finite rank, and if A is hypercentral, i.e., the group generated by an arbitrary element of A and an arbitrary element of G is nilpotent, then A contains a central element of G different from 1. B. I. Plotkin has shown [Dokl. Akad. Nauk SSSR 107 (1956), 648–651; MR 17, 1107] that every nil group with maximality condition is nilpotent. Theorem 4: every nil group with minimality condition is locally nilpotent.

E. R. Kolchin (New York, N.Y.)

3915:

Rédei, Ladislaus. Eine Bemerkung über die endlichen einstufig nichtkommutativen Gruppen. Acta Sci. Math. Szeged 19 (1958), 127–128.

The groups mentioned in the title were studied by the author among others in 1947 [Comment. Math. Helv. 20 (1947), 225–264; MR 9, 131]. Partly, they are not p-groups. Using a number-theoretic theorem of Zsigmondy treated in a second paper [#3845 above] the author now determines the possible normal Sylow groups.

C. G. Lekkerkerker (Amsterdam)

3916:

Chen, Chung-mu. On a theorem of Burnside. Advancement in Math. 4 (1958), 274–276. (Chinese. English summary)

The following theorems are proved: Theorem 1: If a finite group G is p-normal and the normalizer N_P of a p-sylow subgroup P of G is $P\times K$, then there exists a normal subgroup of G whose quotient group is a p-group; Theorem 2: if every element of order relatively prime to p of $N_{\bar{P}}$ is commutative with every element of $\bar{P}$, where $\bar{P}$ is a p-group of G and $N_{\bar{P}}$ is the normalizer of $\bar{P}$, then there exists a normal subgroup $N\neq e$ of G such that $P\cap N=e$, where P is a p-sylow group of G; Theorem 3: if the order of a group G is $n=p^\alpha p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, and $\alpha<\min(p_1, \cdots, p_k)$, and if the normalizer of a p-sylow group P of G is P itself, then there exists a normal subgroup of G whose quotient group is a p-group. *Author's summary*

3917:

Room, T. G.; and Smith, R. J. A generation of the symplectic group. Quart. J. Math. Oxford Ser. (2) 9 (1958), 177–182.

Defining the symplectic group $L_{2m}(G)$ as the group of matrices A having $2m$ rows and columns, such that $A^tGA=G$, where $G=(g_{ij})$ is skew symmetric with $g_{ii}=0$ and $g_{ij}=1$ $(i<j)$, the authors show that $L_{2m}(G)$ can be generated by matrices Q and D^x of specified form which we shall not reproduce here. The form of G was suggested by geometrical considerations.

G. de B. Robinson (Toronto, Ont.)

3918:

Busulini, Bruno. Sulla relazione triangolare in un l-gruppo. Rend. Sem. Mat. Univ. Padova 28 (1958), 68–70.

In a lattice-ordered group G, the familiar triangle inequality $|a+b|\leqq|a|+|b|$ obtains for the absolute value $|a|=a\cup -a$ precisely when G is commutative. Several non-commutative substitutes for the inequality are noted in this paper, $|a+b|\leqq(|a|+|b|)\cup(|b|+|a|)$.

R. S. Pierce (Seattle, Wash.)

3919:

Loonstra, F. Fortsetzung von Gruppenhomomorphismen. J. Reine Angew. Math. 199 (1958), 192–202.

Let A', A, Q be three groups, and k, f, f' three homomorphisms such that $k(A)=A'$, $f(Q)=A$, $f'(Q)=A'$ and $f'=kf$. Let, similarly, B', B, R be groups, and h, d, d' homomorphisms with $h(B)=B'$, $d(R)=B$, $d'(R)=B'$ and $d'=hd$. The paper studies extensions $G'(A'; B')$ (of A' by B'), $G(A; B)$, $S(Q; R)$ and homomorphisms g, t, t' such that: 1) $g(G)=G'$, $t(S)=G$, $t'(S)=G'$; 2) $t'=gt$; 3) g induces k and h, t induces f and d, and t' induces f' and d'. It shows how to construct all such systems of extensions G', G, S and homomorphisms g, t, t' in case A', A, Q are abelian. The main tools are representative systems, factor sets, automorphism groups, kernels of homomorphisms, and cohomology. *T. Nakayama* (Nagoya)

3920:

Berman, S. D. On certain properties of group rings over the field of rational numbers. Užgorod. Gos. Univ. Naučn. Zap. Him. Fiz. Mat. 12 (1955), 88–110. (Russian)

The author considers the group ring over the field R' of rational numbers of a finite abelian group. Through an enumeration of the minimal idempotents, he obtains formulae for the number of irreducible representations of given degree, and from these derives the result, a special case of results published by Perlis and Walker [Trans. Amer. Math. Soc. 68 (1950), 420–426; MR 11, 638], that

two finite abelian groups are isomorphic if their group rings over R' are. He generalizes the theorem to include finitely generated abelian groups, and a class of non-abelian groups including the hamiltonian groups, and produces an example of two non-isomorphic non-abelian groups of order p^3 (prime $p\neq 2$) having isomorphic group rings over R'. *E. R. Kolchin* (New York, N.Y.)

3921:

Roquette, Peter. Realisierung von Darstellungen endlicher nilpotenter Gruppen. Arch. Math. 9 (1958), 241–250.

Suppose that G is a finite group, D an irreducible representation of G with character χ. Denote by Q the field of rational numbers. The author proves that, for a nilpotent group G, D has an equivalent representation with coefficients in the field $Q\{\chi(g),\ g\in G\}=K$, except possibly in case 2 divides the order of G. In the latter case, however, D is certainly equivalent to a representation with elements in $K(\sqrt{-1})$. The procedure of the author involves (i) reduction to faithful primitive representations, (ii) reduction to the Sylow subgroups of G, and (iii) group extensions. Important is the lemma stating that all abelian normal subgroups of a group with a faithful, irreducible, primitive representation module must necessarily be cyclic.

O. F. G. Schilling (Chicago, Ill.)

3922:

Perel, W. M. Principal representations in commutative semigroups. Proc. Amer. Math. Soc. 8 (1957), 957–960.

R. E. MacKenzie [Duke Math. J. 21 (1954), 471–477; MR 16, 8] extended to commutative semigroups the theory of primary decomposition of ideals in rings. The present paper extends MacKenzie's results to show that for suitable families of co-ideals of a semigroup, W. Krull's theory of 'Hauptkomponenten' decompositions of ideals has its analogue for semigroups [Math. Ann. 101 (1929), 729–744]. The analogue of a 'Hauptkomponent' is defined as follows: let S be a member of a family $\mathcal{F}$ of co-ideals of the semigroup $\mathfrak{M}$ [Mackenzie, loc. cit.] and let P be a minimal element of the set of those elements of $\mathcal{F}$ which contain $S{:}S$; then $S{:}P$ is a principal component of S. When applied to a ring the author's results reproduce Krull's results only when the ring is Noetherian.

G. B. Preston (Shrivenham)

3923:

Lyapin, E. S. On the existence and uniqueness of solutions of a general type of equation in connection with reversibility in semigroups of transformations. Dokl. Akad. Nauk SSSR (N.S.) 116 (1957), 552–555. (Russian)

Let $\mathfrak{A}$ be a semigroup. An element S is called left reversible if for any $A\in\mathfrak{A}$ there exists an $X\in\mathfrak{A}$ such that $XS=A$. Similar and obvious definitions apply to right reversible and two-sided reversible elements of A. The author states without proof many elementary properties of reversible elements and their applications to semigroups of transformation on a set Ω. In particular, results concerning the equation $S\xi=\alpha$, for $\alpha\in\Omega$ and S a left (or two-sided) reversible transformation of a semigroup of transformations $\mathfrak{A}$ acting on Ω, are stated without proof. *L. J. Paige* (Los Angeles, Calif.)

3924:

Vagner, V. V. Generalized heaps reducible to generalized groups. Ukrain. Mat. Ž. 8 (1956), 235–253. (Russian)

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See also 3906, 3954, 3955.

3925:

✱Pontrjagin, L. S. Topologische Gruppen. 2. B. G. Teubner Verlagsgesellschaft, Leipzig, 1958. 308 pp. DM 16.00.

This second volume contains the remaining chapters, 6–11, of the translation of "Continuous groups" [2d. ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954; MR 17, 171]. For the first volume of the translation see MR 19, 152.

3926:

Dixmier, Jacques. Quelques propriétés des groupes abéliens localement compacts. Bull. Sci. Math. (2) 81 (1957), 38–48.

The author proves and summarizes results concerning locally compact abelian groups. Let A be a locally compact abelian group. For A to be arcwise connected, it is necessary and sufficient that A be of the type $R^n\times\hat{D}$, where R^n denotes an n-dimensional vector space, D a discrete abelian group with $\mathrm{Ext}_Z^1(D,Z)=0$. For A to be locally arcwise connected, it is necessary and sufficient that A be of the form $R^n\times\hat{D}\times E$, where E is discrete and D is a discrete abelian group such that $\mathrm{Ext}_Z^1(D,Z)=0$ and any subgroup D' with torsion-free D/D' is a direct summand. A is called injective if any representation ρ of a subgroup H can be extended to the whole group. For A to be injective, it is necessary and sufficient that A be of the form $R^n\times T^m$, where T denotes a torus. Concerning A, there are four cases: (a) A is of the form $R^n\times T^m$; (b) A is connected and locally arcwise connected; (c) A is arcwise connected; and (d) A is connected and locally connected. Evidently $(a)\Rightarrow(b)\Rightarrow(c)\Rightarrow(d)$. However, (c)$\neq$(d) in general.

For any discrete abelian group D without torsion, there are also four cases: (a) D is free; (b) $\mathrm{Ext}_Z^1(D,Z)=0$ and every subgroup D' of finite rank with torsion-free D/D' is a direct summand in D; (c) $\mathrm{Ext}_Z^1(D,Z)=0$; and (d) all subgroups of finite rank are free. It is evident that (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d). An example, very similar to that by Specker [Portugal. Math. 9 (1950), 131–140; MR 12, 587] is given to show that (c)$\neq$(d) in general.

{Additional references: Kaplansky, "Infinite abelian groups", Univ. Michigan Press, 1954 [MR 16, 444]; Yamabe, Proc. Japan Acad. 27 (1951), 205–207 [MR 13, 818, 1140.]}

H. Yamabe (Minneapolis, Minn.)

3927:

Teleman, Silviu. Sur la représentation linéaire des groupes topologiques. Ann. Sci. Ecole Norm. Sup. (3) 74 (1957), 319–339.

The author shows that every Hausdorff group is isomorphic to a group of homeomorphisms of a compact space, and also to a group of automorphisms of a Banach algebra of continuous functions on a compact space.

If G is a Hausdorff group, let $\mathcal{F}=\{f_i\}$, $i\in I$, be a saturated family of continuous left invariant pseudo-metrics bounded by 1 on $G\times G$. For each $(x,i)\in G\times I$ define $\varphi_{x,i}(y)=f_i(x,y)$. For each (x,i) let $K_{x,i}=[0,1]$ and consider the product space $K=P_{(x,i)\in G\times I}K_{x,i}$. The canonical map of G into K is one-one and uniformly continuous. Let $\mathcal{U}_{\mathcal{F}}$ be the smallest uniformity on G for which φ is uniformly continous. Then $\mathcal{U}_{\mathcal{F}}\subset\mathcal{U}_l=$ the left uniformity of G, and $\mathcal{U}_{\mathcal{F}}$ is compatible with the topology of G. Let $G_{\mathcal{F}}$ be the compact space which is the completion of G with respect to $\mathcal{U}_{\mathcal{F}}$. Then left translation by members of

G can be extended to $G_{\mathcal{F}}$, because of their uniform continuity, and yields a group Γ of homeomorphisms of $G_{\mathcal{F}}$, algebraically isomorphic to G. Topologize Γ by transporting the topology of G.

For $x \in G_{\mathcal{F}}$ let $H_x:\Gamma \to G_{\mathcal{F}}$ be defined by $H_x(T)=T^{-1}(x)$ for $T \in \Gamma$. Theorem: The left uniformity of Γ is smaller than any uniformity for which the family $\{H_x\}$, $x \in G_{\mathcal{F}}$, is uniformly continuous. Theorem: The right uniform structure of Γ is that of uniform convergence on the compact space $G_{\mathcal{F}}$.

Let $C(G_{\mathcal{F}})$ be the Banach algebra of complex-valued continuous functions on $G_{\mathcal{F}}$. Theorem: Every Hausdorff topological group G is isomorphic to a group Γ of automorphisms of the algebra $C(G_{\mathcal{F}})$, with the left uniformity isomorphic to the trace on Γ of the uniformity of simple convergence on $C(G_{\mathcal{F}})$.

Numerous extensions and applications of these theorems are made, including relations between various compactifications of G. *M. E. Shanks* (Lafayette, Ind.)

3928:

Goetz, A. Invariante Metriken in homogenen Räumen. Fund. Math. **45** (1957), 78–83.

Proof of a result announced earlier [Bull. Acad. Polon. Sci. Cl. III **5** (1957), 139–140; MR **19**, 299]. Furthermore, a necessary and sufficient condition is given for a topological group to have a left invariant uniform structure.

W. T. van Est (Utrecht)

3929:

Montgomery, D.; and Mostow, G. D. Toroid transformation groups on euclidean space. Illinois J. Math. **2** (1958), 459–481.

This paper deals with the action of an r-dimensional toroid G on a space X. The intersection of the isotropy groups G_y, $y \in Y \subset X$, is called the fixer of Y. For $H \subset G$, $H^+(G)$ is the intersection of all the fixers containing H. A point x of X is a weight point if G_x is $(r-1)$-dimensional; the homomorphism $G \to G/G_x$ is the weight associated with x. A fixed point of $H \subset G$ is a point which is fixed under each element of H. The fixed set $F(H)$ of H consists of its fixed points. A star circle subgroup of G is a circle subgroup whose fixed set is not the fixed set of any connected subgroup property containing it. It is shown that if X can be compactified to a cohomology n-sphere over the integers by adding a single point ∞, and if G acts almost effectively, then G is spanned by its star circle subgroups, $0 \leqq \dim F(G) \leqq n-2r$ and $F(G) \cup \infty$ is a cohomology sphere. If G acts effectively and $n=2r$ or $2r+1$, then G has exactly r star circle subgroups; all fixer subgroups are connected; $(ST)^+=S^+T^+$ for any two subgroups S and T; G has exactly r weights, 2^r fixer subgroups and 2^r isotropy classes. In case X is HLC and $n \leqq 2r+1$, it is shown that the set U of points with trivial isotropy subgroups decomposes as $B \times G$, where B is a euclidean-like cohomology manifold.

P. A. Smith (New York, N.Y.)

3930:

Kirillov, A. A. The representations of the group of rotations of an n-dimensional Euclidean space by spherical vector fields. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 538–541. (Russian)

Let S^n be the unit sphere in Euclidean $(n+1)$-space. The tangent vector fields on S^n constitute a linear space $R^{(n)}$. $SO(n+1)$ operates on S^n and, hence, linearly in $R^{(n)}$. The paper exhibits explicitly the irreducible components in $R^{(n)}$ by enumerating their dominant weights. In particular, it is shown that $R_k^{(n)}$ — the subspace of $R^{(n)}$ consisting of the fields whose components are polynomials of degree $\leqq k$ in the coordinates — consists of $2k-1$ irreducible components if $n>3$. Now, $R_k^{(n)}$ contains the products of the fields from $R_1^{(n)}$ with the spherical polynomials of degree $\leqq k-1$, and also the gradients of the spherical polynomials of degree $\leqq k+1$. Again, for $n>3$ one gets, thus, all irreducible components of $R_k^{(n)}$. The cases $n \leqq 3$ are also discussed.

W. T. van Est (Utrecht)

3931:

Saitô, Masahiko. Sur certains groupes de Lie résolubles. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **7** (1957), 1–11.

Let G be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. The author gives a necessary and sufficient condition for the validity of the relation $\exp \mathfrak{g}=G$. The condition is that $\mathfrak{g}$ should not contain any subalgebra isomorphic to $\mathfrak{e}$ or $\mathfrak{d}$. Here, $\mathfrak{e}$ denotes the Lie algebra of the group of rigid motions in the plane and $\mathfrak{d}$ denotes a certain 4-dimensional Lie algebra which is a central extension of $\mathfrak{e}$. When this condition is satisfied the author shows, in addition, that the exponential function maps $\mathfrak{g}$ in a one-to-one fashion onto G and maps the center $\mathfrak{z}$ of $\mathfrak{g}$ onto the center Z of G. In particular, Z is connected. Related results have also been obtained by Dixmier [Bull. Soc. Math. France, **85** (1957), 113–121; MR **19**, 1182].}

S. Helgason (Chicago, Ill.)

3932:

Saitô, Masahiko. Sur certains groupes de Lie résolubles. II. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **7** (1957), 157–168.

Let G be a connected, solvable Lie group with the (real) Lie algebra $\mathfrak{g}$. Let G^* denote the simply connected covering group of G. The author has in a previous paper [#3931 above] given a necessary and sufficient condition that $\mathfrak{g}$ be of type (E), which means that the exponential mapping maps $\mathfrak{g}$ onto G^*. This is here translated into a criterion involving the "roots" of $\mathfrak{g}$. These are linear forms ρ_i on $\mathfrak{g}$ which arise from the action of the adjoint representation on the 1-dimensional factor spaces $\mathfrak{g}_{i-1}/\mathfrak{g}_i$; here $\mathfrak{g}^c \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots$ is a Jordan-Hölder series for the complexification $\mathfrak{g}^c$ of $\mathfrak{g}$. [See Dixmier, Bull. Soc. Math. France **85** (1957), 113–121; MR **19**, 1182; there other criteria for type (E) are also given.] If for every closed subgroup F of G there exists an analytic subgroup H of G such that $F \subset H$ and H/F is compact, it is shown that the roots ρ_i are real-valued (and in particular $\mathfrak{g}$ is of type (E)). The converse holds also, and various other consequences are derived from the assumption that the roots ρ_i are real-valued. For example, the normalizer in G of an analytic subgroup of G is again an analytic subgroup of G, and actually a closed subgroup, due to a theorem of Goto [Math. Japon. **1** (1948), 107–119; MR **10**, 681].

S. Helgason (Chicago, Ill.)

3933:

Takenouchi, Osamu. Sur la facteur-représentation d'un groupe de Lie résoluble de type (E). Math. J. Okayama Univ. **7** (1957), 151–161.

This paper is devoted to the type-problem of the theory of unitary representations of groups. A Lie group is called of type (E) if the exponential maps the Lie algebra onto the group. The main result of this paper is that every unitary representation U of a solvable Lie group of type (E) is of type I, i.e., the ring of operators generated by U is of type I in the sense of Murray and von Neumann. It is also shown that if U is irreducible, then it is obtained from a character of the additive group of real

numbers by repeated formation of induced representations. The proof that U is of type I consists of studying composition series in G (and its Lie algebra) and of finding suitable connected Abelian normal subgroups H of G which are regularly embedded in G. The fact that the representations of G induced by the characters of H are of type I is then used. {This article is closely related to the recent work of Dixmier [see particularly Bull. Soc. Math. France **85** (1957), 113–121; MR **19**, 1182].}

F. I. Mautner (Paris)

3934:

Nôno, Takayuki. On geodesic subspaces of group spaces. J. Sci. Hiroshima Univ. Ser. A. **21** (1957/58), 167–176.

Let G be a connected Lie group with Lie algebra $\mathfrak{G}$ and let exp denote the usual exponential mapping of $\mathfrak{G}$ into G. The author calls a submanifold S of G a geodesic subspace if to each $x \in S$ there corresponds a linear subspace $\mathfrak{S}_x$ of $\mathfrak{G}$ such that the set N_x of elements $x \exp X$, where X runs through a suitable neighborhood of 0 in $\mathfrak{S}_x$, is a neighborhood of x in S. This is a slight generalization of the concept of a totally geodesic submanifold of G. Such manifolds were characterized by E. Cartan in terms of Lie triple systems contained in $\mathfrak{G}$ [J. Math. Pures Appl. **6** (1927), 1–119, especially p. 71]. Without using or referring to Cartan's paper, the author proves that each $\mathfrak{S}_x$ is a Lie triple system and that any two spaces $\mathfrak{S}_x$ and $\mathfrak{S}_y$, where $x, y \in S$, are conjugate under an inner automorphism of $\mathfrak{G}$. The author's proofs are purely group-theoretic, whereas Cartan's proof uses the ordinary torsion-free left invariant linear connection on G in which the geodesics are the cosets of one-parameter subgroups.

S. Helgason (Chicago, Ill.)

FUNCTIONS OF REAL VARIABLES

See also 4030, 4040, 4108, 4109, 4110, 4158.

3935:

Van, Sy-lei. A simple proof of a theorem of Men'šov. Advancement in Math. **3** (1957), 673–674. (Chinese. Russian summary)

The theorem in question is: if $F(x)$ is the upper limit in measure on $[a, b]$ of the sequence of functions $\{f_n(x)\}$, $n=1, 2, \cdots$, which are finite almost everywhere on $[a, b]$, then

$$\limsup_{n\to\infty} f_n(x) \geqq F(x)$$

almost everywhere on $[a, b]$. [For the nomenclature, etc., see, e.g., D. Men'šov, Dokl. Akad. Nauk SSSR **59** (1948), 849–852; MR **9**, 426]. *From the author's summary*

3936:

Lin, Jiing-rong. Functions of bounded variation of order 3. Advancement in Math. **3** (1957), 628–641. (Chinese. English summary)

A real function $f(x)$ defined and bounded on the closed interval $[a, b]$ is said to be of bounded variation of order 3 if, for all subdivisions of $[a, b]$, the quantity

$$\sum_{i=1}^{n-1}\left|\frac{f(x_{i+1})+f(x_i)-2f\left(\frac{x_{i+1}+x_i}{2}\right)}{\left(\frac{x_{i+1}-x_i}{2}\right)^2} - \frac{f(x_i)+f(x_{i-1})-2f\left(\frac{x_i+x_{i-1}}{2}\right)}{\left(\frac{x_i-x_{i-1}}{2}\right)^2}\right|$$

is bounded. Two conditions are given, each of them necessary and sufficient that $f(x)$ be such a function: 1) $f(x)=\int_a^x g(t)dt$, where $g(t)$ is a function of bounded variation of order 2 [see H. Y. Tung, Acad. Sinica Sci. Record **5** (1952), 29–43; MR **15**, 297]; 2) $f(x)$ is the difference of two convex functions of order 2 on $[a, b]$ with finite derivatives $\lim_{h\to 0+} [f(a)+f(a+2h)-2f(a+h)]/h^2$ and $\lim_{h\to 0+} [f(b)+f(b-2h)-2f(b-h)]/h^2$, where by a convex function of order 2 on $[a, b]$ we mean a real function defined on $[a, b]$ satisfying

$$\frac{f(x_2)+f(x_1)-2f\left(\frac{x_1+x_2}{2}\right)}{\left(\frac{x_2-x_1}{2}\right)^2} \leqq \frac{f(x_3)+f(x_2)-2f\left(\frac{x_3+x_2}{2}\right)}{\left(\frac{x_3-x_2}{2}\right)^2}$$

for $a \leqq x_1 < x_2 < x_3 \leqq b$. *From the author's summary*

3937:

Cereteli, O. D. On functions of bounded variation. Soobšč. Akad. Nauk Gruzin. SSR. **19** (1957), no. 2, 129–134. (Russian)

The author uses a new method for studying the structure of functions of bounded variation. This method is based on two original concepts: a so-called condensation function and a scattering function. The condensation function of a measurable set E around a point a is defined as

$$Y_E(y; a) = \begin{cases} \text{meas}[(0, y) \frown E]+a \text{ if } y \geqq 0, \\ -\text{meas}[(y, 0) \frown E]+a \text{ if } y<0. \end{cases}$$

The scattering function $X_E(x; a)$ is the inverse of this function. A number of theorems are proved; a typical one of these is the following result. Let $f(x)$ be a nondecreasing function defined on the interval (p, g). If $f'(x)=1$ almost everywhere on (p, g), then there exists a measurable set E and a number a such that $f(x)=X_E(x; a)$ at every point where $f(x)$ is continuous.

H. P. Thielman (Ames, Iowa)

3938:

Gagaev, B. M. Hyperalgebraic functions. Kazan. Gos. Univ. Uč. Zap. **112** (1952), no. 10, 201–205. (Russian)

3939:

Yang, Tsung-pan. Einige Bemerkungen über die Bairesche Eigenschaft. II. Acta Math. Sinica **8** (1958), 95–101. (Chinese. German summary)

In dieser Note, zeigen wir nochmal einige der Messbarkeit analogen Sätze, betreffend die Bairesche Eigenschaft. Die früher [dieselben Acta **6** (1956), 83–91; MR **18**, 23] eingeführten Begriffe — Bairesche Hülle und Kern — spielen eine Rolle. Bezeichnen wir mit $\mathfrak{B}$ die Gesamtheit der bis auf eine Menge von erster Kategorie endlichwertigen Funktionen, die in $[0, 1]$ definiert sind, und Bairesche Eigenschaft haben; und mit $\mathfrak{N}$ die Gesamtheit der Funktionen, die bis auf eine Menge von erster Kategorie Null sind. Dass der Quotientenraum $\mathfrak{B}/\mathfrak{N}$ ein

stetiger, im Birkhoffschen Sinne bedingt vollständiger, Rieszscher Raum ist, ist bewiesen. Die Beziehung zwischen der Baireschen Eigenschaft einer Funktion und der ihrer Ordinatenmenge, eine Erweiterung des Begriffes Bairescher Eigenschaft ist betrachtet.

Zusammenfassung des Autors

3940:

Froda, Alexandre. Introduction à l'étude des propriétés "à distance" des fonctions réelles. Bull. Sci. Math. (2) **81** (1957), 175–190.

This is a fuller exposition of results announced earlier [C. R. Acad. Sci. Paris **242** (1956), 1948–1951; **243** (1956), 549–552; MR **17**, 953; **18**, 795].

M. E. Shanks (Lafayette, Ind.)

3941:

Froda, Alexandre. Propriétés "à distance" distribuées sur des ensembles relativement denses. Ann. Mat. Pura Appl. (4) **44** (1957), 185–199.

See the preceding review. In this paper the author is primarily concerned with "réseaux" of paraneighborhoods defined on a set E dense in E^n. Here, the theorems of C. R. Acad. Sci. Paris **243** (1956), 549–552 [MR **18**, 795] are proved in full. *M. E. Shanks* (Lafayette, Ind.)

3942:

van der Blij, F. Some consideration concerning the concept of a differential. Simon Stevin **31** (1957), 156–168. (Dutch)

The present article is largely of an expository nature. After emphasising the gap between the notion of differential as introduced in text-books on analysis and as used in applied mathematics, the author discusses the manner in which first and higher order differentials were introduced by Fermat, Leibniz, Cauchy, and more recently by A. Weil, with special reference to differentials on differentiable varieties [Géométrie différentielle, Colloq. Internat. Centre Nat. Rech. Sci., Strasbourg, 1953, pp. 111–117, C.N.R.S., Paris, 1953; MR **15**, 828]. The algebraic point of view is stressed, giving rise to a lucid exposition of the theory of alternating differential forms and the formulation of the integral theorems (Gauss, Stokes) in terms of this calculus. The paper concludes with a brief description of recent work of Kähler [Bericht über die Mathematiker-Tagung, Berlin, 1953, pp. 58–163, Deutscher Verlag der Wissenschaft, Berlin, 1953; MR **16**, 563].

H. Rund (Durban)

3943:

Fenchel, W. On the introduction of the exponential function. Nordisk Mat. Tidskr. **6** (1958), 109–113, 136. (Danish. English summary)

The purpose of this note is to introduce the exponential function in a simple way that is suitable for beginners. The author's treatment is based on the so-called Bernoulli inequality, $(1+h)^n \geqq 1+nh$ for $h \geqq 1$ and n a natural number, proved by induction on n. The existence of $\lim_{n\to\infty}(1+x/n)^n$ and the facts that $e^x e^y = e^{x+y}$ and $de^x/dx = e^x$ are deftly derived, without the use of the binomial theorem or power series.

F. Bagemihl (Notre Dame, Ind.)

3944:

Monroe, M. E. Bringing calculus up-to-date. Amer. Math. Monthly **65** (1958), 81–90.

An interesting discussion of the need to inject more elementary (modern) differential geometry into the undergraduate calculus program. {In this respect, the author seems willing to go further than did the reviewer in his recent text "Advanced Calculus" [McGraw-Hill, New York-Toronto-London, 1956; MR **19**, 732].}

R. C. Buck (Stanford, Calif.)

3945:

Brodovickiĭ, K. V. On the $\int_0^\pi [\sin^m x/(p+q\cos x)]\,dx$ integral. Dokl. Akad. Nauk SSSR **120** (1958), 1178–1179. (Russian)

The integral in the title, for $m>2$, is equal to

$$2^{m-2}\frac{p}{q^2}\sum_{\nu=1}^{k}\left(\frac{p^2-q^2}{-4q^2}\right)^{\nu-1}\mathrm{B}\left(\frac{m+1-2\nu}{2}, \frac{m+1-2\nu}{2}\right) + a\left(\frac{p^2-q^2}{-q^2}\right)^k,$$

where B denotes the beta function, $k=(m-2)/2$ if m is even, $k=(m-1)/2$ if m is odd, $a=\pi pq^{-2}\{1-(1-q^2/p^2)^{\frac{1}{2}}\}$ if m is even, $a=q^{-1}\ln((p+q)/(p-q))$ if m is odd. A formula is also given for $m=2$. These results represent a correction of standard tables of integrals.

3946:

Markus, S. [Marcus, S.] Points of discontinuity and points of differentiability. Rev. Math. Pures Appl. **2** (1957), 471–474. (Russian)

Let $f(x)$ be a finite real valued function on $[0, 1]$. Let D be the set of points of discontinuity and Δ the set of points of differentiability of $f(x)$. The author proves several theorems concerning the sets D and Δ and related sets. Typical are the following. Theorem 1: Let D be everywhere dense in $[0, 1]$. Then Δ is of first Baire category. It is shown by an example that Δ can be of Lebesgue measure one.

Theorem 2: Let $0 \leqq \alpha \leqq 1$. Then there exists a function $f(x)$ for which the set D is everywhere dense and has Lebesgue measure α, and such that the set of points where $f(x)$ has a right derivative equal to ∞ is a residual set.

Theorem 3: Let D be an F_σ set in $(0, 1)$ and let D have measure α. Let $0 \leqq \beta \leqq 1-\alpha$. Then there exists a function $\varphi(x)$ with D as the set of discontinuity of $\varphi(x)$ and such that the set Δ of its points of differentiability is of measure β.

Some of the proofs in this note are based on results of Brudno [Mat. Sb. N.S. **13(55)** (1943), 119–134; MR **7**, 10]. *Y. N. Dowker* (London)

3947:

Beesack, Paul R. Integral inequalities of the Wirtinger type. Duke Math. J. **25** (1958), 477–498.

The basic inequalities discussed in the first part have the general form: If the real functions p and u of x satisfy certain conditions for $a \leqq x \leqq b$, then

$$\int_a^b u'^2 dx \geqq \int_a^b pu^2 dx,$$

with equality when and only when $u=Ay$, where y is a certain solution of $y''+py=0$. The conditions are (i) explicit restrictions, such as p continuous, u an integral (of u'), $u' \in L_2$; (ii) an implicit restriction on p postulating the existence of a y with suitable properties; (iii) boundary conditions at a or b (or both). There are other theorems containing further conditions such as (iv) $\mathfrak{M}(p) \geqq 0$, $\mathfrak{M}(pu)=0$, where $\mathfrak{M}$ denotes a mean value over $a \leqq x \leqq b$; but, as the author observes, these can be derived by changing u to $u-B$ in the results or proofs of basic theorems. The basic inequalities are deduced from the

formal identity

$$(1)\qquad \int_a^b (u'^2-pu^2)dx=\int_a^b (u'-gu)^2dx+[gu^2]_a^b,$$

where $g=y'/y$, so that g satisfies Riccati's equation $g'+g^2=-p$. This is a development of a method already used in proofs of known inequalities for special p [Hardy, Littlewood, and Pólya, Inequalities, Cambridg Univ. Press. 1952; ch. VII; MR 13, 727]. In these proofs the relevance of the equation $y''+py=0$ is recognized from the fact that, with u in place of y, it is Euler's equation for the variational problem of minimizing the integral $J(u)$ on the left of (1); and this is taken as a suggestion for proving the minimizing property of a solution y without explicit use of the calculus of variations. But individual problems have usually been treated separately. The more general and more systematic procedure adopted in this paper achieves a substantial measure of unification. The second part of the paper deals with similar inequalities with u'' in place of u'. Here, the equation for y is $y^{\mathrm{iv}}-py=0$, but the equation for $g=y'/y$ is more complicated and the conditions are correspondingly more elaborate. The paper concludes with a brief comment on an inequality associated with a partial differential equation. {Note by reviewer: The discussion of singularities of g arising from zeros of y seems inadequate. It is asserted that the product $g(u-B)$ is bounded near a common zero of y and $u-B$ (as it would be if y and u were analytic); but a conclusion as strong as this does not follow from general conditions such as those in (i), and is indeed not required. The weaker property $g(u-B)\in L_2$ would suffice, and this can in fact be deduced from the condition assumed.}

A. E. Ingham (Cambridge, England)

3948:

Andersson, Bengt Joel. An inequality for convex functions. Nordisk Mat. Tidskr. 6 (1958), 25–26, 56.

The minimum of

$$\int_0^1 F_1(x)F_2(x)\cdots F_n(x)dx,$$

under the restrictions that each F_p is convex and nonnegative and satisfies $F_p(0)=0$ and $\int_0^1 F_p(x)dx=\alpha_p$, is attained when each F_p is linear.

C. Davis (Providence, R.I.)

3949:

Ciesielski, Z. A note on some inequalities of Jensen's type. Ann. Polon. Math. 4 (1958), 269–274.

Let $f(x)$ be convex in an interval and, for $i=1, \cdots, n$, let x_i be points of the interval and let p_i be positive values. Jensen's inequality is

$$f\left(\frac{\sum_{i=1}^n p_ix_i}{\sum_{i=1}^n p_i}\right)\leq\frac{\sum_{i=1}^n p_if(x_i)}{\sum_{i=1}^n p_i}.$$

Now let I be the closed interval $[0, a]$, let $f(x)$ and $f'(x)$ be convex in I, with $f(0)\leq 0$, let $x_1\geq\cdots\geq x_n$ be points of I, and let p_i be values satisfying $\sum_{i=1}^k p_i\geq 0$ for $k=1, \cdots, n$ and $\sum_{i=1}^n |p_i|>0$. Under these circumstances, the author shows that

$$f\left(\frac{\sum_{i=1}^n p_ix_i}{\sum_{i=1}^n |p_i|}\right)\leq\frac{\sum_{i=1}^n p_if(x_i)}{\sum_{i=1}^n |p_i|}.$$

An analogous result is given for functions of two variables, integral analogues are presented, and applications are made to the establishment of generalizations of inequalities of Chebyshev and Biernacki.

E. F. Beckenbach (Los Angeles, Calif.)

MEASURE AND INTEGRATION

See also 3800, 3801, 4066.

3950:

★Zaanen, Adriaan C. An introduction to the theory of integration. North-Holland Publishing Company, Amsterdam, 1958. ix+254 pp. 25 guilders.

This book is a presentation in textbook form of the modern approach to the theory of integration, not however after the manner of Bourbaki, but based on measure theory of abstract sets in the Caratheodory manner, combined with the general integral concepts due in the first place to P. J. Daniell, and elaborated and extended by M. H. Stone. The main portion of the book is essentially a detailed exposition of the general integration theory due to M. H. Stone [Proc. Nat. Acad. Sci. U.S.A. 34 (1948), 336–342, 447–455, 483–490; 35 (1949), 50–58; MR 10, 24, 107, 239, 360].

The first chapter discusses such widely divergent notions as set theory, Zorn's lemma and its relation to the well ordering theorem, and metric spaces including the Baire category theorem. The second chapter is devoted to the theory of measure. Basic measure is defined on a semi-ring Γ of sets (closed under intersection, and with the property that if B of Γ is contained in A of Γ, then $A-B=\sum C_n$, where C_n are disjoint members of Γ). Such a measure is assumed to be positive, completely additive, with $\mu(Q)=0$. There is the extension of measure to a σ-ring of sets, following Caratheodory. The third chapter is an exposition of the Daniell integral, based on the observation that the integral is essentially an extension of a measure on an ordinate set. The basic integral $I(f)$, defined on a linear vector lattice set of functions L, and subject to the linear, positive property together with monotone convergence to zero under monotone convergence to zero of the functions, gives rise to a measure on a semi-ring of sets, which are differences of sets F and G determined by positive functions f by the condition $F=(x, y)$, for which $0<y\leq f(x)$ if $f(x)<+\infty$ and $0<y<f(x)$ if $f(x)=+\infty$. The proof of the linearity of the resulting general integral, from this point of view, is somewhat messy. Chapter 4 treats Lebesgue-Stieltjes integrals, based on the class L of functions which is the linear extension of the class of characteristic functions, with $I(f)$ defined in the usual way. To connect the measure defined by the Daniell integral with the original measure, the Stone postulate that for any f in L, $\min(f, l)$ is also in L is required. The fifth chapter gives the Fubini double integral theorem as developed by Stone. The sixth chapter, a sidetrack outlining basic notions of linear normed spaces, gives the Banach extension theorem in linear functionals and applies it to the problem of finitely additive measure defined for all point sets on $0\leq x\leq 1$. The notion of Hilbert space is introduced, which leads to the theorem that the most general linear continuous functional on such a space H is of the form (y, x) with y in H. The spaces L_p as linear normed complete spaces are discussed. Chapter 7 is devoted to a proof of the Radon-Nikodym theorem. This follows the Stone presentation which, in turn, is inspired by the procedure of v. Neumann [Ann. of Math. (2) 41 (1940), 124–132; MR 1, 146], which bases the proof on the form of the linear functional in Hilbert spaces. Chapter 8 is an adaptation of ideas of R. de Possel [J. Math. Pures Appl. (9) 15 (1936), 391–409] to the problem of derivatives of set functions. A net is any subdivision of the space X into

a finite or denumerable set of disjoint measurable sets, such nets being ordered by successive subdivisions. A monotone sequence of nets and a set function $\nu(E)$ give rise to sequences of point functions, essentially $\nu(E)/\mu(E)$ for x in E, convergence of which at certain points leads to a derivative notion. By subjecting the sequence of nets N_n to a sort of Vitali condition (called regularity by the author, viz., for every $\varepsilon>0$ and measurable E there exists a sequence of disjoint sets A_m from the nets N_n such that $\mu(E-\sum_m A_m)=0$ and $\mu(\sum_m A_m-E)<\varepsilon$), it is possible to prove both theorems of the metric density type and also the fact that if $\nu(E)=\int_E f d\mu$, then for any regular sequence of nets, the derivative $D\nu(E)$ with respect to μ agrees with f almost everywhere. Chapter 9 is devoted to the introduction of new variables in Lebesgue integrals in n-dimensional Euclidean space, giving results for the case when the transformation $x_i'=f_i(x_1, \cdots, x_n)$, $i=1, \cdots, n$, is nonsingular and has continuous first partial derivatives. Chapter 10 gives a brief application of the general theory developed to integration on the real line. Chapter 11 concerns itself with the extension of results to signed and complex valued measures, including the Jordan and Hahn decomposition theorems. Chapter 12 returns to linear spaces, comments briefly on the Bourbaki approach to the general integration problem, and gives the most general linear continuous forms in L_p spaces, and connected weak convergence. Chapter 13, after a brief discussion of unitary transformation in Hilbert spaces, gives as application the Parseval theorem on Fourier transforms. The final chapter is devoted to a brief discussion of the ergodic theorem in its various aspects. Examples and exercises are distributed liberally throughout the book. Many interesting topics are relegated to the exercises, which are then subjected to strong hints on methods of solution.

T. H. Hildebrandt (Ann Arbor, Mich.)

3951:

Drbohlav, Karel. Eine Bemerkung zur Theorie des Riemannschen Integrals. Časopis Pěst. Mat. **83** (1958), 23–26. (Czech. Russian and German summaries)

The formula $\int_{\phi(\alpha)}^{\phi(\beta)} F(x)dx=\int_\alpha^\beta F(\phi(t))\phi'(t)dt$ is proved if $\phi'\geqq 0$ exists at all points and if $\int_\alpha^\beta \phi'(t)dt$ exists as a Riemann integral. The last hypothesis cannot be replaced by the boundedness of $\phi'(x)$.

František Wolf (Berkeley, Calif.)

3952:

Zaanen, A. C. A note on measure theory. Nieuw Arch. Wisk. (3) **6** (1958), 58–65.

Let X be a non-empty set; Λ a σ-ring of subsets of X; and μ a non-negative countably additive measure on Λ. In most integration-theory contexts the following condition is assumed: every E in Λ such that $\mu(E)$ is positive has a subset F in Λ such that $\mu(F)$ is positive and finite. The author gives three conditions each equivalent to this condition and two conditions which it implies, together with counterexamples to the converse implications. Furthermore, he shows that this condition is equivalent to the equality of the L_∞-norm of an L_∞ function with its norm as a linear functional acting on L_1. Finally, it is equivalent to the following statement for $1<p\leqq\infty$: if f is measurable and fg is integrable for each g in L_q (where $p^{-1}+q^{-1}=1$) then f is in L_p.

W. F. Stinespring (Princeton, N.J.)

3953:

Mickle, Earl J. On a closure property of measurable sets. Proc. Amer. Math. Soc. **9** (1958), 688–689.

S is a function defined over families of subsets of a space X and whose values are also families of subsets of X, such that $F\subset S(F)$ and if $F_1\subset F_2$ then $S(F_1)\subset S(F_2)$, where F, F_1, F_2 are any families of subsets. The author shows that if $F=S(F)$ whenever F is a family of sets measurable with respect to some finite-valued regular outer measure then the same identity holds if F is a family of sets measurable with respect to any outer measure.

H. G. Eggleston (Cambridge, England)

3954:

Mibu, Yoshimichi. Decomposition-equivalence and the existence of non-measurable sets in a locally compact group. Proc. Japan Acad. **34** (1958), 185–188.

The author establishes the existence of non-measurable subsets in a locally compact group G by proving the following results (which can then be applied). Two subsets A, B of a locally compact σ-compact non-discrete group G are called completely decomposition equivalent if

$$A=\bigcup_{i=1}^{\infty} A_i, \quad B=\bigcup_{i=1}^{\infty} g_i A_i,$$

where the sets A_i are disjoint and the sets $g_i A_i$ are also disjoint. If $A-M$, $B-N$ are completely decomposition equivalent, where M, N have Haar measure zero, then A, B are called decomposition equivalent. The sets A_i are not assumed measurable. The author proves that any set of positive measure is decomposition equivalent to G and that any set of non-empty interior is completely decomposition equivalent to G. From this it follows that any locally compact non-discrete G has subsets which are non-measurable.

Interesting subsidiary results include an analogue of the Schröder-Bernstein theorem, that if A is (completely) decomposition equivalent to a subset of itself it is (completely) decomposition equivalent to any intermediate set; also, that any two complete systems of representatives of right cosets of the same countable subgroup H are completely decomposition equivalent.

A. M. Macbeath (Dundee)

3955:

Mibu, Yoshimichi. On homomorphic mappings. Proc. Japan Acad. **34** (1958), 241–244.

Conditions are found for a homomorphism (in the algebraic sense) of a topological group G into a σ-bounded topological group G^* to be continuous, in the following forms: (1) If G is locally compact and m^* is a left-invariant Haar outer measure, a homomorphism is continuous if the inverse image of every open set is m^*-measurable; (2) If all the open sets in G are of the second category, a homomorphism is continuous if the inverse image of every open set has the property of Baire.

The existence of non-measurable sets in a locally compact nondiscrete abelian G is deduced as a consequence. {The author does not seem to know that the first result was proved by Kodaira [Proc. Phys.-Math. Soc. Japan (3) **23** (1941), 67–119; MR **2**, 317].}

A. M. Macbeath (Dundee)

3956:

Kluvánek, Igor. Note on the extension of measure. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **7** (1957), 108–115. (Slovak. Russian and English summaries)

In this article the theorem on extension of measure is proved as follows. Let $\mathbf{R}$ be an algebra of subsets of any set X and μ a totally finite measure on $\mathbf{R}$. Let $\mathbf{R}_\sigma$ [$\mathbf{R}_\delta$] be the class of all sets E such that there exists an increasing [decreasing] sequence $\{E_n\}_{n=1}^\infty$ of sets in $\mathbf{R}$ and $E=\bigcup_{n=1}^\infty E_n$[$E=\bigcap_{n=1}^\infty E_n$]. Denote $\mu_1(E)=\lim_n \mu(E_n)$. Further let $\mathbf{R}_{\sigma\delta}$ [$\mathbf{R}_{\delta\sigma}$] be the class of sets of the form $E=$

$\bigcap_{n=1}^{\infty} E_n$ $[E=\bigcup_{n=1}^{\infty} E_n]$, where $\{E_n\}_{n=1}^{\infty}$ is a decreasing [increasing] sequence of sets in $[\mathbf{R}_\sigma]$ $\mathbf{R}_\delta$ and let $\mu_2(E)=\lim_n \mu_1(E_n)$.

If we denote by $\mathbf{S}$ the system of all sets $E\subset X$ for which there exists a set $A\in\mathbf{R}_{\delta\sigma}$ and a set $B\in\mathbf{R}_{\sigma\delta}$ such that $A\subset E\subset B$ and $\mu_2(A)=\mu_2(B)$, then $\mathbf{S}$ is a σ-algebra containing $\mathbf{R}$ and the function $\bar{\mu}$ on $\mathbf{S}$, unambiguously defined by the equation $\bar{\mu}(E)=\mu_2(A)$, is a (unique) complete measure which coincides with μ on $\mathbf{R}$.

Author's summary

3957:

Kluvánek, Igor. On vector measure. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **7** (1957), 186–192. (Slovak. Russian and English summaries)

In this paper some theorems concerning extension of vector measure are proved.

A vector measure is a σ-additive function μ defined on algebra $\mathbf{R}$ of subsets of an abstract set S, with values in a (real or complex) Banach space X, i.e., if $\{E_n\}$ is any sequence of disjoint sets $\in\mathbf{R}$ and $\bigcup_{n=1}^{\infty} E_n\in\mathbf{R}$, then $\lim_i |\mu(\bigcup_{n=1}^{\infty} E_n)-\sum_{n=1}^{i}\mu(E_n)|=0$.

Let $\mathbf{S}=\mathbf{S}(\mathbf{R})$ be the minimal σ-algebra over $\mathbf{R}$. The following theorems hold. There exists a vector measure $\bar{\mu}$ on $\mathbf{S}$, which is extension of μ (i.e., $\bar{\mu}(E)=\mu(E)$ for $E\in\mathbf{R}$) if and only if there exists a finite non-negative measure ν on $\mathbf{R}$ such that $\lim_{\nu(E)\to 0}\|\mu\|(E)=0$ ($\|\mu\|$ denotes the semivariation of μ, see [1]).

If the space X is reflexive and the semivariation $\|\mu\|$ of μ is finite on $\mathbf{R}$, then there exists a finite non-negative measure ν on $\mathbf{R}$, such that $\lim_{\nu(E)\to 0}\|\mu\|(E)=0$. An example of vector measure is constructed, showing that the requirement of reflexivity of X in the last theorem, must not be omitted.

Author's summary

3958:

✱Halmos, Paul R. Lectures on ergodic theory. Publications of the Mathematical Society of Japan, no. 3. The Mathematical Society of Japan, 1956. vii+99 pp. $2.00.

This book is the first work on ergodic theory in book form since E. Hopf's Ergodentheorie [Springer, Berlin] appeared on 1937. Its contents are based on a course of lectures given by the author at the University of Chicago in 1956. The first of these facts makes the book very welcome; more so since the book is written in the pleasant, relaxed and clear style usually associated with the author. The material is organised very well and painlessly presented. A number of remarks, ranging from the serious to the whimsical, add insight and enjoyment to the reading of the book.

The topics covered are as follows: recurrence, the ergodic theorems, a general discussion of ergodicity and mixing properties, including the two mixing theorems. There is a general discussion of the relation between conjugacy and equivalence. For the particular case of ergodic transformations with a pure point spectrum this relation is discussed together with other properties in some detail. A particularly full and clear account is given of the two theorems concerning the topological sizes of the sets of weak mixing and strong mixing, respectively; the first stating that 'In general a measure preserving transformation is a mixing'; the second, that 'In general a measure preserving transformation is not a mixing'. The book ends with a discussion of the still unsolved problem of the existence of a σ-finite invariant measure equivalent to a given one and a list of some unsolved problems.

The author, in the apology (introduction) to the book, asks the reader to regard these notes as 'designed to rekindle' interest in the subject. From this point of view, and considering the excellent and effortless style of the book, it is doubly regrettable that the material discussed is so restricted in time and person. There is almost no indication of work done during the last decade, and the reviewer cannot but be disappointed that the reader is left unaware of the recent sparks of interest found by workers in such branches of ergodic theory as, for instance, those related to probability theory, number theory, abstract ergodic theory, dynamical systems in general and geodesic flows in particular. *Y. N. Dowker* (London)

3959:

Rivkind, Ya. I. Real functions on Boolean algebras with a measure. Grodnen. Gos. Ped. Inst. Uč. Zap. Ser. Mat. 2 (1957), 89–101. (Russian)

Let $B=B(\Omega,\mu)$ be a Boolean algebra with a measure. A substructure B^* of B is dense in B provided that for each $a\in B$ there exists an $a^*\in B^*$ such that $0\neq a^*\leqq a$ and that every $x\leqq a^*$ belongs to B^*. A function G defined in B^* is called an average function provided $G=\nu/\mu$, where ν is an additive absolutely continuous function with respect to μ. G is ε-constant in $x\in B$, provided the oscillation of G on the set of elements $<x$ is $<\varepsilon$. G is said to have the $(\bar{C})$-property provided that for every $\varepsilon>0$ there exists a dense subset M_ε^* of B such that G be ε-constant in each point of M_ε^*. Suppose that $G(0)=0$. Two functions G_1, G_2 are equivalent provided for each $\varepsilon>0$ there exists a dense subset M_ε^* such that $|G_1(e)-G_2(e)|<\varepsilon(e\in M_\varepsilon^*)$. G_1, G_2 are differentially equivalent on a dense substructure B^* provided for each $e\in B^*$ and each $\varepsilon>0$ there exists a partition $e=\sum_{k=1}^{n} e_k$, $e_k\in B^*$, such that $\sum_{k=1}^{n}|G_1(e_k)-G_2(e_k)|<\varepsilon$. Theorem 1: Average functions coincide with the functions with the $(\bar{C})$-property. The proof is based on this lemma: For functions G_1, G_2, the following propositions are equivalent: (a) G_1 and G_2 are differentially equivalent on a dense substructure B^*; (b) G_1/μ, G_2/μ are equivalent in $B(\Omega,\mu)$. Theorem 2: The equivalence of two average functions ν_1/μ_1, ν_2/μ_2 is equivalent with having each of the relations:

$$\nu_1(e)=\int_e \frac{\nu_2(de)\mu_1(de)}{\mu_2(de)};\ \nu_2(e)=\int_e \frac{\nu_1(de)\mu_2(de)}{\mu_1(de)}$$

hold on a dense substructure. The results are applied in the metrical theory of dynamical systems.

Đ. Kurepa (Zagreb)

3960:

Ramamohana Rao, C. On differentiation under the integral sign. Math. Student **25** (1957), 143–146.

The formula for the derivative of an integral with respect to a parameter occurring in both limits and integrand is established, say, for $y=c$, with the explicit hypotheses by way of continuity being: (i) that the variable limits $a(y)$ and $b(y)$ be continuous in a neighborhood of c and differentiable at c; and (ii) that the integrand $f(x,y)$ be continuous at $(a(c),c)$ and $(b(c),c)$. The other hypotheses are as in the proof for the formula with fixed limits via the Lebesgue dominated convergence theorem (see page 67 in H. Cramer's "Mathematical methods of statistics" [Princeton Univ. Press, Princeton, N.J., 1946; MR **8**, 39]). *G. Crane* (Pittsburgh, Pa.)

3961:

Lee, F. C.; and Chi, Y. S. A sufficient condition of integrability of a sequence of functions. Advancement in Math. 3 (1957), 655–657. (Chinese. English summary)

Let $\{f_n(x)\}$ be a sequence of measurable functions defined on a set E, and let it converge to a function $f(x)$ on E. If there exists a function $\Phi(x)$, positive valued and integrable over E, such that

$$\limsup_{n\to\infty}\sup_{x} |f_n(x)|\cdot mE\{x:|f_n(x)|>\Phi(x)\}=0,$$

then

$$\lim_{n\to\infty}\int_E f_n(x)dx=\int_E f(x)dx.$$

Author's summary

3962:

Deheuvels, R. L'intégrale de Lebesgue. Ann. Inst. Fourier, Grenoble 7 (1957), 383–393.

Given a clan Γ (or ring) of subsets of a fundamental set X, and a measure μ on Γ, the theory of Lebesgue integration for a real valued function f on X is developed by defining the integral for finitely valued functions on X (i.e., $f(x)=c_i$ on a finite number of subsets A_i in Γ of X), introducing the norm $N(f_1-f_2)=\int|f_1-f_2|d\mu$ in this linear space, and completing the resulting space. {That this is possible has, of course, been known a long time.}

T. H. Hildebrandt (Ann Arbor, Mich.)

3963:

Maharam, Dorothy. Automorphisms of products of measure spaces. Proc. Amer. Math. Soc. 9 (1958), 702–707.

If S is the measure theoretic product of a family of unit intervals and ϕ an automorphism of the measure algebra of measurable subsets of S modulo null sets, then the author shows that there is a 1-1 mapping of S onto itself which, with its inverse, preserves measurability, maps null sets onto null sets and induces ϕ. This is an extension of known results [see J. von Neumann, Ann. of Math. 33 (1932), 574–586], in that the number of factors of S is not assumed to be countable nor is it assumed that ϕ is measure preserving.

H. G. Eggleston (Cambridge, England)

3964a:

Nikodým, Otton Martin. On extension of a given finitely additive field-valued, non negative measure, on a finitely additive Boolean tribe, to another tribe more ample. Rend. Sem. Mat. Univ. Padova 26 (1956), 232–327.

3964b:

Nikodym, Otton Martin. On extension of a given finitely additive field-valued measure on a finitely additive Boolean tribe to another one more ample. J. Reine Angew. Math. 199 (1958), 35–52.

Let B and B' be Boolean algebras, $B\subset B'$, and F and F' ordered fields, $F\subset F'$. If μ is an additive F-valued function defined on B, then under certain circumstances μ can be extended to an additive F'-valued function on B'. This proposition is examined in great detail.

E. Hewitt (Seattle, Wash.)

3965:

Bade, William G. A remark on finitely additive measures. Amer. Math. Monthly 65 (1958), 190–191.

On any infinite algebra of sets there exists a non-negative, real-valued, unbounded finitely additive measure. This answers a question asked by the reviewer [Mat. Tidsskr. B 1951, 81–94; MR 14, 964].

E. Hewitt (Seattle, Wash.)

3966:

Sâmboan, G. Sur l'intégrale produit. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 9 (1957), 241–246. (Romanian. Russian and French summaries)

"Pour les fonctions continues $x(\alpha)$: $[a, b]\to X$ où X est une algèbre de Banach avec unité 1, l'intégrale produit de Riemann est définie par

$$\lim_{|\pi|\to 0}\prod_{k=1}^{n}(1+x(\alpha_k)|\Delta_k|)=\int_a^{b}\!\!\!\!\cap\,(1+x(\alpha)d\alpha)$$

[Cf. P. R. Masani, Trans. Amer. Math. Soc. 61 (1947), 147–192; MR 8, 321; p. 151]. La fonction $x(\alpha)$ s'appelle intégrable, s'il existe une suite de fonctions continues sur $[a, b]:\{x_n(\alpha)\}$ avec les propriétés suivantes: elles convergent en mesure vers $x(\alpha)$; $\lim_{m,n}\int_a^b\|x_m(\alpha)-x_n(\alpha)\|d\alpha=0$. Pour une fonction intégrable on définit l'intégrable produit par $\lim_n\int_a^{b\cap}(1+x_n(\alpha)d\alpha)=\int_a^{b\cap}(1+x(\alpha)d\alpha$. Pour les fonctions intégrables et bornées, la formule de Peano pour $x(\alpha)$ demeure encore vraie." (Author's summary)

G. K. Kalisch (Minneapolis, Minn.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 3880, 3887, 4017, 4034, 4092, 4093, 4094, 4163, 4468.

3967:

✠Courant, Richard. Introdução à teoria das funções. [Introduction to the theory of functions.] Translated by L. Barsotti. Sociedade Paranaense de Matemática, Curitiba, 1957. vi+156 pp.

This work is a translation of lecture notes of a course given by R. Courant in New York University. There are 4 chapters: introduction and fundamental concepts, analytic functions, complex integration, power series and analytic continuation.

3968:

Belardinelli, Giuseppe. Sulla risoluzione analitica delle equazioni algebriche generali. Ist. Lombardo Accad. Sci. Lett. Rend. A 92 (1957), 75–96.

Les travaux relatifs à la "résolution analytique" d'une équation algébrique ont été assez rares depuis 30 ans. L'auteur, qui a beaucoup participé au progrès de cette théorie, rappelle qu'elle est fondée sur la considération de la série de Lagrange (développement en puissances de x d'une racine de l'équation $y-w-xf(y)=0$ où $f(y)$ est une fonction analytique) et surtout à la généralisation de cette série au cas de plusieurs variables.

L'auteur rappelle que les coefficients du développement en série d'une racine d'une équation algébrique entière de degré n sont exprimables à l'aide des fonctions hypergéométriques de Pochhammer [Belardinelli, Ann. Mat. Pura Appl. (3) 29 (1920), 251–270; et Birkeland, C. R. Acad. Sci. Paris 171 (1920), 1370–1372; 172 (1921), 309–311]. Partant d'une formule d'inversion de Pincherle-Mellin, l'auteur donne un mode de calcul des coefficients de la série résolvante d'une équation algébrique

$$\theta(y)\equiv(y-\omega_0)(y-\omega_1)\cdots(y-\omega_{n-1})=0,$$

en utilisant les intégrales du type

$$\psi(\alpha,\beta)=\frac{1}{2\pi i}\int_{(\omega_0)} y^{\beta}\theta(y)^{-\alpha}dy.$$

R. Campbell (Caen)

3969:

Kao, Heng-san. A new proof of the fundamental theorem of algebra. Advancement in Math. 3 (1957), 608–611. (Chinese. English summary)

In this paper the author gives a new and elementary proof of the fundamental theorem of algebra. The proof makes use of the Cauchy-Riemann conditions and induction on the degree of the polynomial to show that any polynomial $f(z)=z^n+a_1z^{n-1}+\cdots+a_n$ with complex coefficients whose derivative $f'(z)$ has no double zeros has a zero. If $f'(z)$ has double zeros, then the resultant $R(f', f'')=0$. The author then constructs a sequence of polynomials $f_m(z)=z^n+\alpha_1^{(m)}z^{n-1}+\cdots+\alpha_n^{(m)}$; $m=1, 2, \cdots$, such that $R(f_m', f_m'')\neq 0$ and $\alpha_i^{(m)}$ converges to a_i for each i. Since $f_m(z)$ converges uniformly to $f(z)$ on any bounded domain, the first result is then extended to $f(z)$.

S. Lin (Columbus, Ohio)

3970:

Han, Khwat Tik; and Kuipers, L. Some remarks on the Cauchy index theorem. Tôhoku Math. J. (2) 9 (1957), 238–242.

Let $f(z)=A(z-a_1)\cdots(z-a_m)$ and $g(z)=B(z-b_1)\cdots(z-b_n)$, $AB\neq 0$, denote two real polynomials without common real zeros. Let $\lambda=\lambda_1+\lambda_2 i$, λ_1 and λ_2 real and $\lambda_2\neq 0$, and let $F(z)=f(z)+\lambda g(z)$. By use of the Principle of Argument, the numbers μ_1 and μ_2 of zeros of $F(z)$ in the half planes $\mathrm{Im}(z)>0$ and $\mathrm{Im}(z)<0$, respectively, are determined by the authors as follows. Arrange in increasing order the real numbers in the set $a_1, a_2, \cdots, a_m$; $b_1, b_2, \cdots, b_n$; from each subset of $2p$ or $2p+1$ consecutive a_i, delete $2p$ a_i; and make like deletions from the consecutive b_j. This operation is to be repeated until no consecutive a_i or consecutive b_j remain. The resulting set is denoted by S. Let k be the number of surviving a_i in S if $m\geqq n$ or the number of surviving b_j in S if $m\leqq n$. Let $\gamma=\lambda_2(\alpha_1-\beta_1)g(\alpha_1)f(\beta_1)$ if S contains both some a_i and some b_j, of which α_1 and β_1 denote the smallest. Let $\gamma=-\lambda_2 g(\alpha_1)$ if S contains only an $a_i=\alpha_1$ and let $\gamma=\lambda_2 f(\beta_1)$ if S contains only a $b_j=\beta_1$. Let $\gamma=0$ if $k=0$. Then the authors find that $\mu_r=(\frac{1}{2})$ $[\max(m, n)-(-1)^r k \operatorname{sgn}\gamma]$, $r=1, 2$. When $\lambda=i$, this result reduces to the Cauchy index theorem. Various additional theorems are derived from this result.

M. Marden (Milwaukee, Wis.)

3971:

Ahmad, Salah. Sur la probabilité pour qu'une série entière à coefficients aléatoires puisse être prolongée. C. R. Acad. Sci. Paris 246 (1958), 2574–2576.

Denote by E the set of power series (convergent in a given circle C) which cannot be continued analytically across the boundary and by e the set of those power series convergent in C which can be continued across C. Polya [Acta. Math. 41 (1917), 99–118] has shown that, in a convenient topology defined on the set $E\cup e$, the set E is everywhere dense and has only interior points, while the set e consists only of isolated elements. Frechet [C. R. Acad. Sci. Paris 165 (1917), 359–360] has shown that in another topology defined on $E\cup e$ one can obtain the results of Polya with E and e interchanged. Steinhaus [Math. Z. 31 (1929), 408–416] has given a stochastic interpretation of this problem and has shown that if (φ_n) is a sequence of independent random variables uniformly distributed on the interval $[0, 2\pi]$, then the series $\sum e^{i\varphi_n} z^n$ admits the circle of convergence as a cut with probability one. The present author continues along the direction of the work of Steinhaus and proves that if (φ_n) is a sequence of independent random variables such that $|Ee^{i\varphi_n}|\leqq\alpha<1$, $n=0, 1, \cdots$, then $\sum e^{i\varphi_n}z^n$ admits the circle of convergence as a cut with probability one.

V. F. Cowling (Lexington, Ky.)

3972:

Čibrikova, L. I.; and Rogožin, V.S. Reduction of certain boundary problems to a generalized Riemannian problem. Kazan. Gos. Univ. Uč. Zap. 112 (1952), no. 10, 123–127. (Russian)

3973:

Čibrikova, L. I. A special case of the generalized Riemann problem. Kazan. Gos. Univ. Uč. Zap. 112 (1952), no. 10, 129–154. (Russian)

3974:

Ganin, M. P. The Riemann boundary value problem for a system of functions. Uspehi Mat. Nauk (N.S) 13 (1958), no. 3(81), 173–177. (Russian)

The problem of the title is: Given a boundary L made up of a finite set of disjoint, simple, closed, smooth curves in the complex plane, and given a function $A(t)$ defined from L into the set of n-by-n matrices, L bounds a connected region S^+ and another region S^-; find a vector-valued function $\varphi=(\varphi_1, \cdots, \varphi_n)$ such that each φ_k is analytic in S^+ and in S^- and such that the boundary values $\varphi^+(t)$ and $\varphi^-(t)$ from the two regions satisfy $\varphi^+(t)=A(t)\varphi^-(t)$. Some results are stated for special forms of the matrix function $A(t)$.

M. M. Day (Urbana, Ill.)

3975:

Wang, Chen-yu. Some properties of holomorphic functions. Advancement in Math. 3 (1957), 612–622. (Chinese. English summary)

Five theorems are proved concerning bounds for $|f^{(n)}(z)|$, where $f(z)$ is analytic in the unit circle and satisfies

$$\iint_{|z|<r}|f(z)|^p dxdy=M<\infty,$$

where p is a fixed positive number and M is independent of r.

From the author's summary

3976:

Wang, Hung-schen. Ein anderer Beweis des Satzes von Warschawski. Advancement in Math. 4 (1958), 304–305. (Chinese. German summary)

A concise proof of the theorem in Warschawski's article on the boundary behavior of the derivative of the mapping function in a conformal mapping [Math. Z. 35 (1932), 321–456].

3977:

Kubo, Tadao. Kelvin principle and some inequalities in the theory of functions. III. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 29 (1955), 119–129.

The author continues previous investigations [same Mem. 28 (1953), 299–311; 29 (1955), 17–26; MR 16, 122, 914] on the application of the Kelvin minimum energy principle. Several inequalities concerning the solutions of mixed boundary value problems for harmonic functions are derived. These results are applied, e.g., to prove the monotonicity of certain domain functionals pertaining to conformal slit mapping problems.

O. Lehto (Helsinki)

3978:

Unkelbach, Helmut. Über Ecken mit imaginären Winkeln und ihre konforme Abbildung. Monatsh. Math. 62 (1958), 200–211.

The paper continues a study of the conformal mapping of simply connected regions on a Riemann surface without branch points whose boundary projects onto a finite set of circular arcs [Math. Ann. 129 (1955), 391–414; MR 17, 143]. The author has a definition of angles for the vertices of such a generalized polygon which is based on the Schwarzian derivative of the mapping onto a half-plane. This definition is extended to cases not covered before, and applications are made to locally schlicht meromorphic functions. *P. R. Garabedian* (Stanford, Calif.)

3979:

Radojčić, M. Über die Weierstrasssche Produktentwicklung analytischer Funktionen auf Riemannschen Flächen. Ann. Acad. Sci. Fenn. Ser. A. I, no. 250/27 (1958), 11 pp.

The author treats some theorems pertaining to the possibility of expansions of the Weierstrass product type for analytic functions on non-compact Riemann surfaces. A comparison is made between the author's work and cognate results [notably those of H. Florack, Schr. Math. Inst. Univ. Münster (1948), no. 1; MR 12, 251]. *M. H. Heins* (Urbana, Ill.)

3980:

Stoilow, S. Sur la théorie topologique des recouvrements riemanniens. Ann. Acad. Sci. Fenn. Ser. A. I, no. 250/35 (1958), 7 pp.

A discussion is given of the role played by the general theory of interior transformations in the theory of Riemann surfaces. Particular attention is paid to normally exhaustible coverings and coverings having the Iversen property. In addition, some remarks are made concerning recent work on interior maps into the projective plane. *M. H. Heins* (Urbana, Ill.)

3981:

Read, A. H. Conjugate extremal problems of class $p=1$. Ann. Acad. Sci. Fenn. Ser. A. I, no. 250/28 (1958), 8 pp.

Let W be a compact sub-region of a Riemann surface, bounded by a finite number of Jordan curves γ, and let g_0 and g_1 be functions defined on γ, with $g_1\neq 0$ and continuous there. The author studies sup $|g|$ amongst functions g on γ such that g_1g-g_0 represents the boundary values of some analytic function over W. He also studies $\max|\int_\gamma g_0\omega|$ amongst the analytic differentials ω in W with $\int_\gamma |g_1\omega|\leqq 1$. By means of the representation of linear functionals in Banach space and the Hahn-Banach theorem, it is shown that both of these extremal problems have solutions and that the usual relation $\sup|g|=\max|\int_\gamma g_0\omega|$ holds. *P. R. Garabedian* (Stanford, Calif.)

3982:

Linden, C. N. The minimum modulus of functions regular and of finite order in the unit circle. Quart. J. Math. Oxford Ser. (2) 7 (1956), 196–216.

Let $M(r;f)$ and $\mu(r;f)$ be, respectively, the maximum and minimum of $|f(z)|$ on $|z|=r$. The general question which is the concern of this paper is to find how large $\mu(r)$ can be, compared with $M(r)$. For an entire function f of finite order ρ, it is known that there is a constant C, depending only on ρ, such that $\mu(r)\geqq O(1)/M(r)^C$ for an increasing sequence of r values. This was sharpened to the form $\limsup_{n\to\infty}\log\mu(r)/\log M(r)\geqq-(2.19)\log\rho$ by Hayman, who also showed [Proc. London Math. Soc. (3) 2 (1952), 469–512; MR 15, 22] that for functions f of infinite order, one had

$$\limsup_{n\to\infty}\log\mu(r)/\log M(r)\log\log\log M(r)\geqq-2.19.$$

The author of the present paper treats the analogous problem for functions g analytic in the unit circle. One says that g is of order at most β if $\log M(r;g)\leqq O(1)/(1-r)^{\beta'}$ for all $\beta'>\beta$. There is then a constant $K(\beta)$ such that if g is any function of order $\beta>1$,

$$\limsup_{r\to 1}\log\mu(r)/\log M(r)\log\log M(r)\geqq-K.$$

Examples are given to show that the factor $\log\log M(r)$ cannot be deleted. It is conjectured that $K(\beta)\to\infty$ as $\beta\to 1$, and an example is given of a function g of order 1 and without zeros for which $\log\mu(r)/\log M(r)\log\log M(r)\to-\infty$. *R. C. Buck* (Stanford, Calif.)

3983:

Clunie, J. Inequalities for integral functions. Quart. J. Math. Oxford Ser. (2) 9 (1958), 1–7.

Let $f(z)$ be an entire function with $M(r,f)=\max|f(z)|$ $(|z|=r)$, and $M_n(r,f)=\max|f^{(n)}(z)|$ $(|z|=r)$. Let $\chi(r)$ be a real function with continuous nth derivative for all $r>r_0\geqq 0$. If $M(r,f)\leqq\chi(r)$ for all $r>r_0$ then, for every $k>1$, there exists a sequence $r_p\to\infty$ $(p=1,2,\cdots)$ such that $M_n(r_p,f)<k\chi^{(n)}(r_p)$. *A. G. Azpeitia* (Amherst, Mass.)

3984:

Krishnamurthy, V. A theorem on the zeros of entire functions of exponential type. Proc. Amer. Math. Soc. 9 (1958), 300–304.

The author proposes as his main result the following extension of a theorem by the reviewer [Trans. Amer. Math. Soc. 83 (1956), 417–429; MR 18, 471] that corresponds to the case $N=1$, $\theta_1=0$ below. He supposes given N lines $\theta=\theta_k$, $0\leqq\theta_k<\pi/2$, $k=1,\cdots,N$, and N sequences $\beta_{k,n}=r_{k,n}e^{i\theta_k}$, $n=1,2,\cdots$, with each $r_{k,n}$ a positive integer, $r_{k,n+1}>r_{k,n}$, and puts $\Delta(k)=\limsup_{t\to\infty}A_k(t)/t$, where $A_k(t)$ is the number of $r_{k,n}$ not exceeding t.

Let F be the class of entire functions $f(z)$ that satisfy $f(z)=O(1)\exp(\gamma|z|)$ for some $\gamma<\infty$, $f(iy)=O(1)\exp(c|y|)$ for some $c<\pi(\cos\theta_1+\cdots+\cos\theta_n)$ and $f(\beta_{n,k})=0$ for $k=1,\cdots,N$ and $n=1,2,\cdots$. Theorem: If $\Delta(k)=1$ for $k=1,\cdots,N$, then F contains only the function $f(z)\equiv 0$. Conversely, if $\Delta(k_0)<1$ for at least one k_0, then F contains an $f(z)\not\equiv 0$.

By overlooking the fact that $\cos 2\theta$ is negative if $\pi/4<\theta<\pi/2$, the author introduces several demonstrably false inequalities (p. 302) and invalidates his proof of the "conversely" part unless the restriction $\theta_{k_0}\leqq\pi/4$ is added. The methods are straightforward adaptations of those used in the paper cited above. Considerably deeper methods, not yet published, due to P. Malliavin and the reviewer, jointly, can be used to show that the theorem is true as stated. *L. A. Rubel* (Urbana, Ill.)

3985:

Bose, S. K.; et Srivastav, R. P. Certaines propriétés de la fonction maximum d'une fonction méromorphe. Ann. Sci. École Norm. Sup. (3) 75 (1958), 37–47.

S. K. Bose has defined [Bull. Calcutta Math. Soc. 44 (1952), 69–74; MR 15, 23] a "maximum function" $S(r)$ for a meromorphic function $f(z)=f_1(z)/P(z)$, where $f_1(z)$ is entire and $P(z)$ is a canonical product, as $S(r)=M_1(r)/\mathscr{M}_2(r)$, where $M_1(r)$ is the maximum modulus of $f_1(z)$ and $\mathscr{M}_2(r)$ is the minimum modulus of $P(z)$ outside a set of circles of finite total length surrounding the poles.

In this paper the two authors have attempted to prove results on $S(r)$ and $S^{(1)}(r)$, the "maximum function" of $f^{(1)}(z)$. The first theorem is that

$$\limsup_{r\to\infty} \frac{\log\left\{r\dfrac{S^{(1)}(r)}{S(r)}\right\}}{\log r}=\rho$$

(a result analogous to one proved by the reviewer for entire functions with $M(r)$ and $M^{(1)}(r)$ instead of $S(r)$ and $S^{(1)}(r)$ [Bull. Amer. Math. Soc. 53 (1947), 1156–1163; MR 9, 342]), provided that there exist points w and w^1 of the z-plane such that (*) $|f(w)|=S(r)$ and $|f^{(1)}(w^1)|=S^{(1)}(r)$.

The authors do not say anything about these points w and w^1, but presumably $|w|=r=|w^1|$. The proof of the theorem is not complete; it depends on a result to be proved in another paper by R. P. Srivastav. If $|w|=r=|w^1|$ in (*), then the hypothesis of the theorem is obviously not satisfied by a class of meromorphic functions of order ρ (consider, for instance, $0\leq\rho<1$, $f(z)=\prod(1+z/a_i)/\prod(1+z/b_\delta)$, where $a_i>0$, $b_\delta>0$) and the authors have not discussed the class of functions which satisfy the conditions of theorems 1, 2, 3, 5. A number of inequalities, similar to those proved by the reviewer for entire functions with $M(r)$, $M^{(1)}(r)$, $\cdots$ in the paper mentioned above, have been given with additional hypotheses of the type (*). *S. M. Shah* (Madison, Wis.)

3986:

Singh, S. K.; and Dwivedi, S. H. The distribution of a-points of an entire function. Proc. Amer. Math. Soc. 9 (1958), 562–568.

Let $f(z)$ be an entire function of positive finite order ρ and lower order λ; let $\rho(r)$ be a proximate order and $\lambda(r)$ a lower proximate order [Shah, J. London Math. Soc. 12 (1948), 31–32; MR 10, 441]. The authors prove the following theorems. (1) If $\limsup r^{-\lambda(r)}\log M(r)<\infty$ and $r^{-\lambda(r)}N(r,a)\to0$, for $x\neq a$ we have $0<\liminf r^{-\lambda(r)}N(r,x)\leq1$ and $0<\phi(\lambda)\leq\limsup r^{-\lambda(r)}N(r,x)\leq\lambda^{-1}\limsup r^{-\lambda(r)}n(r,x)<\infty$ with an explicit $\phi(\lambda)$. (2) $\liminf r^{-\lambda(r)}n(r)\leq\lambda$ and $\liminf r^{-\rho(r)}n(r)\leq\rho$. (3) If $\lim N(r,x)/r^{\lambda(r)}=a$, then $n(r,x)/r^{\lambda(r)}\to\lambda a$; if $\lim N(r,x)/r^{\rho(r)}=A$, then $n(r,x)/r^{\rho(r)}\to\rho A$. (4) If $n(r,a)/\log M(r)\to0$, then $\liminf N(r,a)/\log M(r)=0$. (5) If $\liminf \log M(r)/r^{\rho(r)}>0$ and $\lim N(r,a)r^{\rho(r)}=0$, then for $x\neq a$ we have $0<\liminf N(r,x)/r^{\rho(r)}\leq\limsup N(r,x)/r^{\rho(r)}\leq1$. Finally, although $\limsup r^{-\rho(r)}n(r,a)<\infty$ for all a, we cannot replace $\rho(r)$ by $\lambda(r)$ here. *R. P. Boas, Jr.* (Evanston, Ill.)

3987:

Shankar, Hari. On the characteristic function of a meromorphic function. I. Tôhoku Math. J. (2) 9 (1957), 243–246.

Meromorphic functions $f(z)$ of order ρ $(0<\rho<\infty)$ in $|z|<\infty$ are considered. Let $L(x)$ be a positive real valued function defined for $x\geq x_0$ (>0) and such that $L(cx)\sim L(x)$ $(x\to\infty)$ for any positive c. Using standard notations $T(r)$ and $S(r)$ [R. Nevanlinna, Eindeutige analytische Funktionen, 2te Aufl., Springer, Berlin-Göttingen-Heidelberg, 1953; MR 15, 208], let $\tau=\limsup_{r\to\infty}T(r)r^{-\rho}L(r)^{-1}$ and $\mu=\limsup_{r\to\infty}S(r)r^{-\rho}L(r)^{-1}$. Let t and ν be defined analogously by replacing lim sup by lim inf.

The author proves that

$$\nu\leq\rho t\leq\nu(1+\log(\mu/\nu))\leq\mu \quad \text{if } \nu\neq0,$$
$$\nu\leq\mu e^{-1}e^{\nu/\mu}\leq\rho\tau\leq\mu\leq e\rho\tau.$$

K. Oikawa (Los Angeles, Calif.)

3988:

Lohwater, A. J. The boundary behavior of meromorphic functions. Ann. Acad. Sci. Fenn. Ser. A. I, no. 250/22 (1958), 6 pp.

By using topological methods, the author proves: Let $w=f(z)$ be analytic and bounded in $|z|<1$, and let the modulus $|f(re^{i\theta})|$ have radial limit 1 for all $e^{i\theta}$ on an arc $A[\vartheta_1<\vartheta<\vartheta_2]$ of $|z|=1$, except possibly for a set of capacity zero. Then the range $R(f,P)$ of $f(z)$ at each singularity P of $f(z)$ on A is the set $|w|<1$ with at most one exception. This theorem provides, as an application, an improvement of a result of the reviewer [Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 398–401; MR 17, 143]. *K. Noshiro* (Nagoya)

3989:

Bagemihl, F. On power series, area, and length. Michigan Math. J. 4 (1957), 281–283.

Let $\mathfrak{L}_p$ $(p\geq1)$ denote the Banach space of all sequences of complex numbers $\{a_k\}$ for which $\sum|a_k|^p<\infty$. With each element of $\mathfrak{L}_p$ is associated a power series $f(z)=\sum a_kz^k$, which is holomorphic in $|z|<1$.

A convex region D in $|z|<1$ is called a tangential domain if the intersection of its boundary with the circle $|z|=1$ is the single point $z=1$ and if every line through $z=1$ (except the tangent to the circle) intersects D. Let D_θ denote the result of rotating D around the origin through the angle θ. The first theorem of the present paper asserts that, given any tangential domain D, those elements of $\mathfrak{L}_1$ whose associated power series map every region D_θ, $0\leq\theta<2\pi$, into a Riemann configuration of infinite area form a residual subset of $\mathfrak{L}_1$.

The second result of the paper is concerned with the mapping of rectilinear segments S in $|z|<1$ one of whose endpoints lies on $|z|=1$. It is shown that, for any $p>1$, those elements of $\mathfrak{L}_p$ whose associated power series maps every such segment S into a curve of infinite length form a residual subset of $\mathfrak{L}_p$. *F. Herzog* (East Lansing, Mich.)

3990:

Tumarkin, G. C.; and Havinson, S. Ya. Existence in multiply-connected regions of single-valued analytic functions with a given modulus of boundary values. Izv. Akad. Nauk SSSR Ser. Mat. 22 (1958), 543–562. (Russian)

In the n-ply connected domain G with rectifiable Jordan boundary Γ let the function $F(z)$ be analytic and many-valued, possess a single-valued modulus, and be free of branch points. Then the principal theorem of this paper, proved by methods of functional analysis, states that points $z_1, z_2, \cdots, z_m$, $m\leq n-1$, can be found in G such that the function $F^*(z)=F(z)\exp\{-\sum_{k=1}^m[g(z,z_k)-ih(z,z_k)]\}$ will be single-valued in G, where $g(z,\zeta)$ is the Green's function for G and $h(z,\zeta)$ is its conjugate; the modulus of the exponential factor on Γ is one. This theorem is then used in establishing a number of results; in particular, the following generalization and refinement of a theorem due to O. Lokki [Ann. Acad. Sci. Fenn. Ser. A. I. Math.-Phys. no. 144 (1952); MR 14, 742]: (*) If the real valued function $\rho(\zeta)\geq0$ is such that

$$\int_\Gamma |\ln\rho(\zeta)|\frac{\partial g}{\partial n}\,ds<+\infty,$$

then there exists in G a single-valued analytic function $f^*(z)$, having in G no more than $n-1$ zeros, such that on Γ, almost everywhere, $|f^*(\zeta)|=\rho(\zeta)$. Moreover,

$$\ln|f^*(z)|=\frac{1}{2\pi}\int_\Gamma \ln\rho(\zeta)\frac{\partial g(\zeta,z)}{\partial n}\,ds-\sum_{k=1}^m g(z,z_k),$$

and f^* belongs to class D, ($f(z) \in D$ if in G, $U^M(z) \to 0$ as $M \to \infty$, where $M > 0$ and U^M is the best harmonic majorant of the subharmonic function $\ln^+ |f(z)/M|$). The paper is concluded with a restatement of (*) valid for non-rectifiable Γ. *J. F. Heyda* (Cincinnati, Ohio)

3991:

Bazilevič, I. E. Regions of the initial coefficients of bounded univalent functions with p-fold symmetry. Mat. Sb. N.S. **43(85)** (1957), 409–428. (Russian)

Let $S_p(M)$ denote the class of functions $f(z) = z + \sum_{k=1}^{\infty} c_{pk+1} z^{pk+1}$ which are regular and univalent and for which $|f(z)| \leq M$ in $|z| < 1$, with p a positive integer, and set $c_{pk+1} = a_{pk+1} + ib_{pk+1}$. The author determines the domain of variation for the point P (a_{p+1}, a_{2p+1}) for $f(z) \in S_p(M)$ with $b_{p+1} = b_{2p+1} = 0$. Further, he determines the domain of variation for the point $P(|c_{p+1}|, |c_{2p+1}|)$ for the class $S_p(M)$. Similar precise results are obtained for three other classes naturally related to the class $S_p(M)$.

All of his results are obtained using the Loewner parametric representation for the coefficients of a univalent function. *A. W. Goodman* (Lexington, Ky.)

3992:

Waadeland, Haakon. Ein Golusin'scher Satz über schlichte Abbildungen von $|\zeta| > 1$. Norske Vid. Selsk. Forh., Trondheim, **30** (1957), 165–167.

Let $w = \zeta(1 + \alpha_n \zeta^{-n} + \alpha_{n+1} \zeta^{-n-1} + \cdots)$ be schlicht with $|w| > \rho$ in $|\zeta| > 1$, and let $n \geq 2$, $n \leq p \leq 2n-1$. It is shown that $|\alpha_p| \leq 2p^{-1}(1-\rho^p)$. The proof is based on a consideration of the successive iterates $w_1 = \rho w[\rho^{-1} w(\zeta)]$, $w_2 = \rho^2 w_1[\rho^{-2} w_1(\zeta)]$, $\cdots$, which permit a reduction to the case $\rho = 0$ treated by Golusin in 1938. The extremal regions are circles of radius ρ with p symmetrically located radial slits. *P. R. Garabedian* (Stanford, Calif.)

3993:

Waadeland, Haakon. Über ein Koeffizientenproblem für schlichte Abbildungen des $|\zeta| > 1$. Norske Vid. Selsk. Forh., Trondheim, **30** (1957), 168–170.

Let $w = \zeta + c_1 \zeta^{-1} + c_2 \zeta^{-2} + \cdots$ be schlicht in $|\zeta| > 1$, and let $C_n = \max |c_n|$. It is shown that

$$C_{2k-1} \geq k^{-1}\{1 + 2\exp[-2(k+1)/(k-1)]\}.$$

The proof depends on consideration of $f(\zeta^{-k})^{-1/k}$, where f is an appropriate extremal function for the coefficient region of a_2 and a_3 in the class of normalized schlicht functions in the unit circle.

P. R. Garabedian (Stanford, Calif.)

3994:

Hornich, Hans. Zur Struktur der schlichten Funktionen. II. Abh. Math. Sem. Univ. Hamburg **22** (1958), 176–179.

In a previous paper [same Abh. **22** (1958), 38–49; MR **20** #980], the author had investigated the structure of the set of functions univalent in the unit circle by introducing a certain metric in the space R of all functions regular in $|z| < 1$ and vanishing at the origin. He showed, among other things, that the connected components K of R are the sets of functions $f(z) + g(z)$, where $f(z)$ is a fixed function of R and $g(z)$ runs over all entire functions which vanish at the origin. Another result of the above-mentioned paper was the fact that, in every component K of R, the subset of non-univalent functions is itself connected.

In the present paper the author continues this investigation by giving an example of a component K of R in which the subset of univalent functions is not connected. For this purpose he chooses that function $f(z) \in R$ which is defined by $f'(z) = \log[(1-z)/(1+z)]$ (with $\log 1$ chosen as 0). It is shown that $f(z) + \lambda z$ is univalent in $|z| < 1$ if and only if $|\Im\lambda| \geq \pi/2$. From this it can be concluded that, for any fixed $\gamma(z)$ ($\gamma(z)$ entire, $\gamma(0) = \gamma'(0) = 0$), the set of functions $f(z) + \lambda z + \gamma(z)$, where λ runs over all complex numbers, is not connected. Finally, the functions $f(z) + i\pi z$ and $f(z) - i\pi z$ cannot belong to the same connected set of univalent functions in R.

F. Herzog (East Lansing, Mich.)

3995:

Abe, Hitoshi. On conformal mapping of a ring-shaped domain. Sûgaku **8** (1956/57), 25–27. (Japanese)

Let D be the image of $0 < r < |z| < 1$ under a conformal mapping. We assume that the respective images C_r and C_1 of $|z| = r$ and $|z| = 1$ are analytic Jordan curves, and that C_r lies in the domain bounded by C_1. Denote by $L(\rho)$ the length of the image of $|z| = \rho$, $r \leq \rho \leq 1$, and by $A(\rho)$ the area enclosed by that image. The author proves that $L(\rho)/\rho$ is increasing, that $2(1-r^2)A(1) \geq L(1) - L(r)/r \geq 2(1-r^2)A(r)/r^2$ and that $A(1) - A(r) \geq (1-r^2)L^2(r)/(4\pi r^2)$. If, in addition, C_r and C_1 are convex curves, then $A(1) - A(r) < L^2(1)\log(1/r)/\pi$. *M. Ohtsuka* (Hiroshima)

3996:

Robertson, M. S. Schlicht Dirichlet series. Canad. J. Math. **10** (1958), 161–176.

Let $\{f(z)\}$ be a family of regular functions in D; then $f(z)$ is called uniformly locally univalent in D, if for every closed domain $D^* \subset D$ and for every $z_0 \in D^*$, there exists a positive number ρ, which is independent of z_0 and $f(z)$, such that $|z - z_0| < \rho$ belongs to D and $f(z)$ is univalent in $|z - z_0| < \rho$.

Let $f(z) = -e^{-\lambda_1 s} + \sum_{n=2}^{\infty} a_n e^{-\lambda_n s}$ $(s = \sigma + it)$ be a Dirichlet series, whose abscissa of absolute convergence is $\bar{\sigma}$ $(-\infty \leq \bar{\sigma} < \infty)$; then there exists a smallest real number τ $(\bar{\sigma} \leq \tau < \infty)$ for which $\sum_{n=2}^{\infty} \lambda_n |a_n| e^{-\lambda_n \tau} \leq 1$. We call such $f(z)$ a function of class τ. The author proves that $f(z)$ of class τ is uniformly locally univalent in a half-plane. If $\tau < (\log \lambda_1)/\lambda_1$, this half-plane is the one for which $\Re s > \tau$ and if $\tau \geq (\log \lambda_1)/\lambda_1$, the half-plane is the one for which $\Re s > \dfrac{\lambda_q \tau - \log \lambda_1}{\lambda_q - \lambda_1}$, where q is the suffix of the first non-vanishing a_q; and the theorem is a best possible one.

M. Tsuji (Tokyo)

3997:

Alenicyn, Yu. E. On functions without common values and the outer boundary of the domain of values of a function. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 1055–1057. (Russian)

The author states nine theorems without proof. Let B be a finitely-connected domain not containing $z = \infty$, let $a \in B$ and let $R_a(B) = \{f(z)|$ regular, $|f(z)| < 1$ for $z \in B$, $f(a) = 0\}$. Let $F(z, a)$ be a function in $R_a(B)$ such that $f(z) \in R_a(B)$ implies $|f'(a)| \leq F'(a, a)$. Theorem. If $f_1(z)$ and $f_2(z)$ are meromorphic and have no common values in B then

$$|f_1'(z_1) f_2'(z_2)| \leq |f_1(z_1) - f_2(z_2)|^2 F'(z_1, z_1) F'(z_2, z_2).$$

Let $\mathfrak{M}(a_1, a_2) = \{(f_1(z), f_2(z)) | f_1(0) = a_1, f_2(0) = a_2, f_1$ and f_2 meromorphic and without common values in $|z| < 1\}$, and let $\mathfrak{M}_s(a_1, a_2)$ be the subset for which both functions are univalent. Lemma. Let $\mathscr{F}(w_1, w_2)$ be a real valued function. If one of the bounds

$$M_1(|z_1|, |z_2|) \leq \mathscr{F}(f_1(z_1), f_2(z_2)) \leq M_2(|z_1|, |z_2|)$$

is valid for the class $\mathfrak{M}_s(a_1, a_2)$, then it is valid in the class

$\mathfrak{M}(a_1, a_2)$. Here, M_1 is decreasing and M_2 is increasing in both variables. A similar result is stated for the ring $q<|z|<1$.

The lemma is applied to the class $\mathfrak{M}(0, \infty)$ to obtain a theorem from which one can deduce a number of known inequalities about Bieberbach-Eilenberg functions [see Jenkins, Trans. Amer. Math. Soc. **76** (1954), 389–396; MR **16**, 24.] *A. W. Goodman* (Lexington, Ky.)

3998:

Komatu, Yûsaku. On analytic functions with positive real part in a circle. Kōdai Math. Sem. Rep. **10** (1958), 64–83.

Let R be the class of regular analytic functions $\Phi(z)$ defined in $|z|<1$ for which $\operatorname{Re}\Phi(z)>0$ and $\Phi(0)=1$. The author proves the following generalization of a theorem by Rogosinski [Math. Z. **35** (1932), 93–121]. Let $\mathscr{L}$ be a linear operator defined on R such that $\mathscr{L}[\Phi(z)]$ is an analytic function in $|z|<r$ $(r<1)$ which is assumed to satisfy certain other requirements. Let F be a bounded, increasing, convex function defined on the range of $|\mathscr{L}|$. Then

$$\int_{-\pi}^{\pi} F(|\mathscr{L}[\Phi(re^{i\theta})]|)d\theta \leq \int_{-\pi}^{\pi} F(|\mathscr{L}[(1+re^{i\theta})/(1-re^{i\theta})]|)d\theta;$$

and $\Phi(z)=(1+\varepsilon z)/(1-\varepsilon z)$, $|\varepsilon|=1$, is always an extremal for this inequality. By specializing $\mathscr{L}$ and F, he obtains estimates for such things as the area of the image of $|z|<r$ $(r<1)$ and the length of the image of a radial segment under functions in R.

G. Springer (Lawrence, Kan.)

3999:

Komatu, Yûsaku. On analytic functions with positive real part in an annulus. Kōdai Math. Sem. Rep. **10** (1958), 84–100.

Let $R=\{\Phi(z)\}$ denote the family of analytic functions which are regular and single valued in the annulus $q<|z|<1$, with $\operatorname{Re}\Phi(z)>0$, $\operatorname{Re}\Phi(z)=1$ on $|z|=q$, and $\int_{-\pi}^{\pi}\operatorname{Im}\Phi(qe^{i\theta})d\theta=0$. The author proves an inequality similar to that stated in the preceding review [#3998], but with the right side replaced by $\int_{-\pi}^{\pi}F(|\mathscr{L}[\Phi^*(re^{i\theta})]|)d\theta$, where $\Phi^*(z)=(2/i)(\zeta(i\log z)-(\eta_1 i\log z)/\pi)$ and ζ and η_1 have their usual meaning, as in the Weierstrass theory of elliptic functions. In this problem $\Phi^*(\varepsilon z)$ is always an extremal. By specializing $\mathscr{L}$ and F, applications are made to finding estimates for such things as the length of the image of $|z|=r$, $q<r<1$, the area of the image of $r_1<|z|<r_2$, and the length of the image of a radial segment under functions in R.

G. Springer (Lawrence, Kans.)

4000:

Baker, Irvine Noel. Zusammensetzungen ganzer Funktionen. Math. Z. **69** (1958), 121–163.

In the first part of this article the author investigates the growth of functions of the form $f(g)$, where f and g are both entire functions of low finite order. The following are typical results. (A) Let f and g be two entire transcendental functions of order less than $\frac{1}{2}$, and let L be an arbitrary positive constant. Then there exists a sequence of closed Jordan curves Γ_i containing the origin in their interiors, and such that $\bar{\sigma}_i=\max(|z|, z\in\Gamma_i)\to\infty$, for which $|f(g(z))|>\exp(\bar{\sigma}_i^L)$ for $z\in\Gamma_i$. (B) Let constants $0<\rho_f<1$, and $0<\rho_g<1$, and a real valued function $\mu(r)$, with $\lim_{r\to\infty}\mu(r)=\infty$, be given. Then there exist two entire functions f and g of order ρ_f and ρ_g, respectively, such that

$$\max_{|z|=r}|f(g(z))|<\exp(r^{\mu(r)})$$

for all r sufficiently large.

The second part of the paper is devoted to a study of the permutability of entire functions. Among the problems considered are: Given f entire and transcendental, to determine all entire g such that $f(g)=g(f)$. It is known that all positive integral iterates of f are solutions; however, in some cases there are also other solutions. A partial answer is given in the following theorem of the author. If the non-constant polynomial g is permutable with the entire transcendental function f, then g must be of the form $ze^{2\pi im/n}+b$, where m and n are positive integers. Conversely, given a g of this form, there always exists an entire transcendental function f which is permutable with g. If $f=ae^{bz}+c$, $a\neq0$, $b\neq0$, then a g that permutes with f either is a constant (a fixed point of f), or $\equiv z$, or is a positive integral iterate of f.

Another problem treated by the author is the characterization of entire functions F for which there exist two entire transcendental functions f and g such that $F=f(g)=g(f)$. For the case where the additional assumption $f=g$ is made, the following result is proved: If f satisfies the condition

$$\limsup_{r\to\infty}\frac{\log\log\log\max_{|z|=r}|f(f(z))|}{\log r}\leq A,\quad 0\leq A<\infty,$$

then $f(f)$ can have at most $2[2A]$ ($[x]$ is the integral part of x) different asymptotic values. (Here, asymptotic values are considered as different only if they actually differ in numerical value, not if they are assumed along different paths of determination.) The paper concludes with a proof of the result that there does not exist an f holomorphic for $|z|<\delta$ such that $f(f(z))=e^z-1$. Throughout the article, there are ample references to results obtained in this field previously.

W. J. Thron (Boulder, Colo.)

4001:

Arsove, M. G. Proper Pincherle bases in the space of entire functions. Quart. J. Math. Oxford Ser. (2) **9** (1958), 40–54.

Let $\alpha_n(z)=z^n\{1+\lambda_n(z)\}$, $n=0, 1, \cdots$, be entire functions with $\lambda_n(0)=0$. The author is concerned with the question of whether every entire function can be expressed uniquely in the form $\sum c_n\alpha_n(z)$, uniformly convergent on compact sets. After a lengthy introduction, the author invokes a general theorem of Boas to show (as did Boas himself!) that a sufficient condition is that $\lambda_n(z)\to0$. {Boas [Trans. Amer. Math. Soc. **48** (1940), 467–487; MR **2**, 80], in discussing this special case, had pointed out that the result had been obtained earlier by Takahashi [Tôhoku Math. J. **35** (1932), 242–243] and Sheffer [Amer. J. Math. **57** (1935), 587–614; see also R. C. Buck, Lectures on functions of a complex variable, pp. 409–419, Univ. of Mich. Press, Ann Arbor, 1955 [MR **17**, 140].} The only new result in the present paper is in the last section. It is well known that a basic set of polynomials $\{p_n\}$ must obey degree $(p_n)\geqq n$ for infinitely many n; the author shows that this remains true if a finite number of the p_n are not polynomials. *R. C. Buck* (Stanford, Calif.)

4002:

Leont'ev, A. F. On sequences of linear aggregates formed from solutions of differential equations. Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 201–242. (Russian)

The article deals with sequences of linear aggregates

consisting of solutions $y_k(z, \lambda_n)$ ($k=1, 2, \cdots, S$, $n=1, 2, \cdots$) of the differential equations

$$Q_0(z)y^{(s)}+\cdots+Q_s(z)y=\lambda_n y, \quad \limsup_{n\to\infty} \frac{n}{|\lambda_n|^{1/s}}<\infty,$$

where $Q_0(z), \cdots, Q_s(z)$ are certain analytic functions. A number of well-known properties of Taylor's series, Dirichlet's series and sequences of Dirichlet polynomials may be carried over to these sequences.

Author's summary

4003:

Pokalo, A. K. Summation of series of functions of $B^{(r)}$ classes. Minsk. Gos. Ped. Inst. A. M. Gor'k. Uč. Zap. **7** (1957), 35–49. (Russian)

The author considers means $A_n=\sum_{k=0}^{n}\mu_n(k)c_kz^k$ of functions $f(z)=\sum c_kz^k$ regular in $|z|<1$ and satisfying $|f^{(r)}(z)|\leqq 1$ in $|z|<1$. By writing

$$\mu_n(k)=\sum_{j=0}^{p}\binom{k}{\nu}\Delta^\nu\mu_n(0)+\rho_p(n,k),$$

A_n is expressed in terms of the first $r-1$ derivations of $f(z)$, a partial sum of the Taylor Series of $z^n f^{(r)}(z)$ and an error term which can be estimated for many choices of the $\mu_n(k)$ (e.g., Cesàro, Bernstein-Rogosinski, de la Vallée Poussin methods). The results are used to find $\mu_n(k)$ such that $|A_n(z)-f(z)|$ is of the order of the best Chebychev approximation of $f(z)$ by polynomials of degree n.

W. H. J. Fuchs (Ithaca, N.Y.)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

4004:

Kobayashi, Shoshichi; and Nomizu, Katsumi. On automorphisms of a Kählerian structure. Nagoya Math. J. **11** (1957), 115–124.

A Kählerian manifold M is called non-degenerate if the (restricted homogeneous) holonomy group σ_x of the underlying Riemannian manifold contains the endomorphism I_x of the tangent space defining the complex structure (if this is the case at a point x, it is so at all points). The main result states that for a non-degenerate complete Kählerian manifold M the largest connected group $A^0(M)$ of affine transformations of M consists of automorphisms, i.e., of isometries preserving the complex structure. The proof is based on the following facts. (1) The analogue for Kählerian manifolds of the de Rham decomposition; if the manifold is non-degenerate, there is no Euclidean factor and the irreducible Kählerian factors are non-degenerate. (2) If M is an irreducible and non-degenerate Kählerian manifold, the group $A^0(M)$ preserves the complex structure. (3) The known result [cf. Hano, same J. **9** (1955), 99–109; MR **17**, 891; and Kobayashi, ibid. **9** (1955), 39–41; MR **17**, 892] that, for a complete irreducible Riemannian manifold, $A^0(M)$ consists of isometries. — Relations are given between the Ricci curvature and the non-degeneracy of a Kählerian manifold. As a corollary of the methods used in the paper it is shown that on a $2n$-dimensional simply connected Riemannian manifold there exist 0, 2 or continuously many Kählerian structures (i.e., complex structures for which the metric is Kählerian) according to whether the holonomy group (a) is not contained in a real representation of $U(n)$, (b) is contained in a real representation of $U(n)$ but not of $Sp(l)$, $n=2l$, or (c) is contained in a real representation of $Sp(l)$, $n=2l$.

B. Eckmann (Zurich)

SPECIAL FUNCTIONS

See also 4143.

4005:

Morduhaĭ-Boltovskoĭ, D. D. An example of a pseudo-elliptic integral. Rostov. Gos. Ped. Inst. Uč. Zap. **4** (1957), 31–33. (Russian)

We determine under what condition the integral $\int(x+A)\{(x+a)(x+b)(x+c)(x+d)\}^{-\frac{1}{2}}dx$ is pseudo-elliptic, that is, can be expressed in terms of elementary transcendental functions.

From the introduction

4006:

Wintner, Aurel. On Heaviside's generalizations of the exponential function. J. Math. Phys. **37** (1958), 143–146.

The function considered is $e_\alpha(z)=\sum_0^\infty z^n/\Gamma(n+\alpha)$, $\alpha>0$. The author describes properties of e^x which e_α possesses. Monotonicity and sign on $(-\infty,\infty)$ are investigated. Recursion formulas, definite integrals, and an asymptotic expansion are desired.

N. D. Kazarinoff (Ann Arbor, Mich.)

4007:

✱Bowman, Frank. Introduction to Bessel functions. Dover Publications Inc., New York, 1958. x+135 pp. $1.35.

Unabridged and unaltered republication of the 1938 edition, published by Longmans, Green and Co., London-New York.

It contains definitions of Bessel functions and modified Bessel functions of any real order, together with formulas for their computation and relations between them. Some applications are given.

4008:

van Rossum, H. Systems of orthogonal polynomials connected with Bessel functions. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 366–375.

The power series (A) $J_{v+1}(x^{\frac{1}{2}})/2x^{\frac{1}{2}}J_v(x^{\frac{1}{2}})=s_{0,v}+s_{1,v}x+s_{2,v}x^2+\cdots$, where $J_v(x)$ is the series for the Bessel function of real order v, is studied here. (1) It is shown that the Padé table for the series (A) is normal. (2) Explicit expressions are given for the numerators and denominators of the Padé fractions $P_{n,n}(x)/Q_{n,n}(x)$ and $P_{n+1,n}(x)/Q_{n+1,n}(x)$ for (A) in terms of the functions ${}_2F_3$. They are thus related to the Lommel polynomials $R_{n,v}(x)=(\Gamma(v+n)/\Gamma(v))(\frac{1}{2}x)^{-n}{}_2F_3(-\frac{1}{2}n, \frac{1}{2}(-n+1), v, -n, 1-v-n; -x^2)$. (3) Several systems of orthogonal polynomials connected with (A) are shown, for example, the polynomials $C_n(x, v)\equiv 4(v+1)(\Gamma(v+1)x^{-\frac{1}{2}}/2^{2n+3}\Gamma(v+2n+3))R_{2n+1,v+2}(x^{-\frac{1}{2}})=x^n{}_2F_3(-n,-\frac{1}{2}-n; v+2,-2n-1,-v-2n-2; -x^{-1})$. The recurrence relations, weight functions, and orthogonality relations are determined.

E. Frank (Chicago, Ill.)

4009:

Ragab, F. M. On an identity involving Bessel polynomials. Illinois J. Math. **2** (1958), 236–239.

The identity

$$\sum_{r=0}^{n} {}^nC_r(1-a-2k-n)_r\gamma_{k+n-r}(x, a+r, b)(b/x)^r=\gamma_k(x, a, b)$$

is shown to follow from the known recursion relations for

Bessel polynomials. An integral involving a modified Bessel function is then evaluated, and several other identities are established on the basis of these results.

N. D. Kazarinoff (Ann Arbor, Mich.)

4010:

Felsen, L. B. Some new transform theorems involving Legendre functions. J. Math. Phys. **37** (1958), 188–191.

The transform theorems associated with the spectral theorem for the characteristic Green's function $G_\theta(\theta, \theta'; x, \lambda)$ defined by $\{z(\sin\theta)z-(x^2+\frac{1}{4})\sin\theta-\lambda/\sin\theta\}G_\theta=-\delta(\theta-\theta')$, $\theta_1\leqq\theta\leqq\theta_2$, $z=d/d\theta$ are investigated under various boundary conditions at θ_1 and θ_2. G_θ arises in diffraction problems for cones.

N. D. Kazarinoff (Ann Arbor, Mich.)

4011:

MacRobert, T. M. Integrals involving E-functions. Math. Z. **69** (1958), 234–236.

The author evaluates two integrals involving E-functions [an account of the E-functions will be found in the author's 'Functions of a complex variable', 4th edition, Macmillan, London, 1954; for a review of 3rd edition, see MR 9, 20]. In one of the integrals the argument of the E-function contains a factor $\lambda^{-m}(1-\lambda)^{-n}$, where λ is the variable of integration and m and n are positive integers. In the other integral m and n are replaced by $-m$ and $-n$.

A. Edrei (Syracuse, N.Y.)

4012:

Ragab, F. M. Expansions for products of two Whittaker functions. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-23** (1957), i+13 pp.

The author evaluates some series involving products of Whittaker functions by using well-known integral representations and interchanging summation and integration.

P. Henrici (Los Angeles, Calif.)

4013:

Ragab, F. M. Some formulas for the products of E-functions and Whittaker functions. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-24** (1957), i+8 pp.

A p-fold integral involving the product of p Whittaker functions of argument x_i and a Whittaker function of argument $b/\prod x_i$ is evaluated in terms of another Whittaker function. *P. Henrici* (Los Angeles, Calif.)

4014:

Ragab, F. M. Some formulae for the product of Bessel and Legendre functions. Math. Z. **68** (1958), 338–339.

Evaluation of an integral involving the product of a hypergeometric function of argument x and a MacRobert E-function of argument $c(1+x)^{-1}$ between the limits 0 and ∞. *P. Henrici* (Los Angeles, Calif.)

4015:

Ragab, F. M. Integration of E-functions and related functions with respect to their parameters. Nederl. Akad. Wetensch. Proc. Ser. A 61 = Indag. Math. **20** (1958), 335–340.

Integrals of E-functions with respect to their parameters are evaluated. Many specializations to the case of Bessel and various hypergeometric functions are considered. *N. D. Kazarinoff* (Ann Arbor, Mich.)

4016:

Ragab, F. M. An expansion involving confluent hypergeometric functions. Nieuw Arch. Wisk. (3) **6** (1958), 52–54.

The formula

$$\frac{\Gamma(\beta)\Gamma(\alpha-\beta-\gamma)}{\Gamma(\alpha-\gamma)\Gamma(\beta+\gamma)}E(\alpha-\gamma, \beta+\gamma::z)= \sum_0^\infty \frac{(\gamma;r)z^{-2r}}{\Gamma(\alpha+r)\Gamma(\gamma+r)r!}E(\alpha+r,\ \beta+r::z) \times E(\gamma+r,\ \alpha-\beta-\gamma+r::z),$$

$z\neq0$, $|\arg z|<\pi$, is established using an integral representation for E. This E-function is essentially a Whittaker function. *N. D. Kazarinoff* (Ann Arbor, Mich.)

4017:

Frank, Evelyn. A new class of continued fraction expansions for the ratios of Heine functions. Trans. Amer. Math. Soc. **88** (1958), 288–300.

Heine en 1846 et 1847 a publié plusieurs études sur la série qui porte son nom:

$$\phi(a, b, c, q, z)=1+\frac{(1-e^{qa})(1-e^{qb})}{(1-e^{q})(1-e^{qc})}z + \frac{(1-e^{qa})(1-e^{q(a+1)})(1-e^{qb})(1-e^{q(b+1)})}{(1-e^{q})(1-e^{2q})(1-e^{qc})(1-e^{q(c+1)})}z^2+\cdots,$$

dans lesquelles il donne, en particulier, un développement du quotient $\phi(a, b, c, q, z)/\phi(a, b+1, c+1, q, z)$ en fraction continue sous la forme

$$1+\frac{a_1z}{1+}\,\frac{a_2z}{1+}\,\frac{a_3z}{1+}\cdots$$

sur son domaine de convergence. La présente étude propose un autre développement en fraction continue. Il est cette fois de la forme

$$1+\frac{d_1z}{f_1z+1+}\,\frac{d_2z}{f_2z+1+}\cdots,$$

et est obtenue par application d'identités dues à Heine.

Des développements analogues sont obtenus pour les quotients analogues de fonctions de Heine voisines, par exemple, de celles qu'on obtient en conservant le même numérateur et en prenant pour dénominateurs successivement:

$$\phi(a+1, b, c+1, q, z),\quad \phi(a+1, b, c, q, z),$$
$$\phi(a, b+1, c, q, z),\quad \phi(a, b, c+1, q, z)$$
$$\phi(a+1, b+1, c+1, q, z),$$

puis à d'autres quotients encore analogues où z, au numérateur, est remplacé par qz.

Pour terminer, une étuide approfondie règle la question de la convergence de tous ces développements, et des cas particuliers sont étudiés pour des valeurs remarquables des paramètres. *R. Campbell* (Caen)

4018:

Sharma, K. C. A theorem on Meijer transform and infinite integrals involving G-function and Bessel functions. Proc. Nat. Inst. Sci. India. Part A **24** (1958), 113–120.

La transformée de Meijer de $f(x)$ est

$$\Phi(p)=(2/\pi)^{\frac{1}{2}}p\int_0^\infty (px)^{\frac{1}{2}}K_\nu(px)f(x)dx.$$

(Elle se réduit à celle de Laplace si $\nu=\pm\frac{1}{2}$.)

L'auteur donne un théorème (d'expression compliquée) relatif à la représentation de $\phi(p)$ par une intégrale définie portant sur des fonctions hypergéométriques.

Une application est donnée au calcul de certaines intégrales définies (d'expression compliquée) portant sur des fonctions K_ν et des fonctions W_{km} de Whittaker.

Les calculs s'appuyent sur des résultats donnés par Erdélyi [Erdélyi, Magnus, Oberhettinger et Tricomi, Higher transcendental functions, McGraw-Hill, New York-Toronto-London, 1953; MR **15**, 419].

R. Campbell (Caen)

4019:

Melvin, M. A.; and Swamy, N. V. V. J. Evaluation of certain physically interesting integrals and hypergeometric sums. J. Math. Phys. **36** (1957), 157–163.

It is known that the integral

$$\int_0^\infty z^p M_{\alpha,\beta}(z) M_{\gamma,\delta}(z)\,dz,$$

if convergent, can be expressed in terms of a hypergeometric function of two variables [A. Erdélyi et al., Tables of integral transforms. Vol. 1, McGraw-Hill, New York-Toronto-London, 1954; MR **15**, 868; p. 216]. Using known identities the authors express this result for certain special values of the parameters in terms of a terminating ${}_3F_2$.

P. Henrici (Los Angeles, Calif.)

4020:

Brafman, F. A generating function for associated Legendre polynomials. Quart. J. Math. Oxford Ser. (2) **8** (1957), 81–83.

The following result is proved.

$$(*)\quad \rho^{-1}\left(\frac{1+t+\rho}{1-t+\rho}\right)\frac{k!k!}{(1+\alpha)_k(1-\alpha)_k}\,P_k^{(\alpha,-\alpha)}\left(\frac{\phi-tu}{\rho}\right)$$
$$\times P_k^{(-\alpha,\alpha)}\left(\frac{\phi+tu}{\rho}\right)=\sum_{n=0}^\infty P_n^{(\alpha,-\alpha)}(x)\,{}_3F_2\left[\begin{matrix}-k,\,k+1,\,-n;\,u\\ 1+\alpha,\,1-\alpha\end{matrix}\right]t^n$$

where $\rho=(1-2x+t^2)^{\frac12}$, $\rho=1$ for $t=0$; $\phi=(\rho^2+2tux-2t^2u+u^2t^2)^{\frac12}$, $\phi=1$ for $t=0$; and $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial. For $\alpha=0$, (*) reduces to a result obtained by S. O. Rice [Duke Math. J. **6** (1940), 108–119; MR **1**, 234]. The proof of (*) makes use of Bailey's reduction formula for the Appell function F_4 [W. N. Bailey, Generalized hypergeometric series, Cambridge Univ. Press, London, 1935; p. 81, (1)].

L. Carlitz (Durham, N.C.)

4021:

Carlitz, L. The product of certain polynomials analogous to the Hermite polynomials. Amer. Math. Monthly **64** (1957), 723–725.

Define polynomials $H_{p,m}(x)$ by the formula

$$\exp(pxt-t^p)=\sum_{n=0}^\infty H_{p,m}(x)t^m/m!.$$

Thus, the $H_m(x)=H_{2,m}(x)$ are the Hermite polynomials and satisfy the classical identity

$$H_m(x)H_n(x)=\sum_{r=0}^{\min(m,n)} 2^r r!\binom{m}{r}\binom{n}{r}H_{m+n-2r}(x).$$

The reviewer [same Monthly **64** (1957), 89–91; MR **18**, 570] gave a short proof of this by reduction to a simple algebraic relation, and noted an analogous algebraic relation; the author points out that this latter relation similarly implies

$$\phi_m(x)\phi_n(x)=\sum_{k=0}^{\min(m,n,[(m+n)/3])} 3^k k!\binom{m}{k}\binom{n}{k}\frac{(m+n-2k)!}{(m+n-3k)!}\phi_{m+n-3k}(x),$$

where $\phi_m(x)=H_{3,m}(x)$. Extending a method of G. N. Watson [J. London Math. Soc. **13** (1938), 29–32], he presents a second proof of this, and, in analogy with another classical identity for Hermite polynomials, proves that

$$\phi_{m+n}(x)=m!n!\sum_{r,s}(-3)^{r+s}\phi_{m-2r-s}(x)\phi_{n-r-2s}(x)$$
$$\times\{r!s!(m-2r-s)!(n-r-2s)!\}^{-1}.$$

M. P. Drazin (Baltimore, Md.)

4022:

Carlitz, L. The bilinear generating function for Hermite polynomials in several variables. Math. Z. **68** (1957), 284–289.

Elegant proofs, using a partial differential operator argument, of several results on multi-dimensional Hermite polynomials, including the bilinear generating function due to A. Erdélyi [Math. Z. **44** (1939), 201–211].

P. Henrici (Los Angeles, Calif.)

4023:

Chihara, T. S. Nonlinear recurrence relations for classical orthogonal polynomials. Amer. Math. Monthly **65** (1958), 195–197.

The author obtains a non-linear recurrence of the type

$$(*)\quad X^2(p_n'^2-p_np_n'')=U_np_n^2-\frac{\beta_n^2}{C_n}p_{n-1}p_{n+1}+\beta_nV_np_np_{n-1},$$

where p_n denotes one of the classical orthogonal polynomials. For $p_n(x)=H_n(x)$, $L_n^{(\alpha)}(x)$ or $C_n^\lambda(x)$, (*) reduces to formulas in a paper by Webster [same Monthly **64** (1957), 249–252; MR **19**, 28; see also Al-Salam (ibid. **64** (1957), 29–32; MR **18**, 570]. In particular, the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is characterized by (*).

L. Carlitz (Durham, N.C.)

4024:

Chak, A. M. A generalization of Lommel polynomials. Duke Math. J. **25** (1957), 73–82.

The polynomials $R_{p,m,n}(x)$ studied by the author are related to Humbert's generalization of the ordinary Bessel function (a suitably normalized hypergeometric series of type ${}_0F_2$) as the classical Lommel polynomials $R_{m,n}(x)$ are related to the ordinary Bessel function. The author establishes several recurrence relations and determinant identities involving the $R_{p,m,n}(x)$. In a special case he establishes a limit relation similar to one for $R_{m,n}(x)$ which enabled Hurwitz to discuss the reality of the zeros of the ordinary Bessel function.

P. Henrici (Los Angeles, Calif.)

4025:

Bhonsle, B. R. On the integro-exponential functions, $E_\nu(x)$. Bull. Calcutta Math. Soc. **49** (1957), 157–162.

The author derives a number of integral formulas for

$$E_n(x)=\int_1^\infty t^{-n}e^{-xt}dt.$$

Although the results "are believed to be new", some are due to Busbridge (as the author remarks), others are paraphrases (in changed notation) of known results, and at least (3.4) appears to be incorrect as it stands. In the last section the author defines an integral transformation with kernel $E_n(px)$, remarks that it is included in the transformation investigated by R. S. Varma, and writes down a few formal properties of this transformation.

A. Erdélyi (Pasadena, Calif.)

4026:

Ham, Frank S. Expansions of the irregular Coulomb function. Quart. Appl. Math. **15** (1957), 31–39.

Convergent series are obtained for the solutions of

$$(1)\qquad \frac{d^2U}{dr^2}+\left[-\frac{1}{n^2}+\frac{2}{r}-\frac{L(L+1)}{r^2}\right]U=0$$

for non-integral values of $2L+1$ in the forms

$$(2)\qquad J_{\pm(2L+1)}{}^{n}(z)=\sum_{k=0}^{\infty} n^{-2k}\sum_{q=2k}^{3k}\frac{a_{k,q}{}^{(L)}}{b_{k,q}{}^{(L)}}\left(\frac{z}{2}\right)^{q} J_{(2L+1)+q}(z)$$

(J_m=Bessel function). A method for obtaining the numerical coefficients $a_{k,q}$, $b_{k,q}$ is outlined. It is shown in Appendix I that these series converge absolutely and uniformly for $|z|\leqq R$, $|n|\geqq n_0$, R and n_0 arbitrary >0. A second solution of (1) may be obtained, which is independent of $J_{2L+1}{}^{n}(z)$ for all values of $2L+1$ (including $2L+1$=positive integer), by the limit form

$$N_{2L+1}{}^{n}(z)=$$

$$\lim_{M\to L}\left[\frac{\Gamma(n+M+1)}{n^{2M+1}\Gamma(n-M)}J_{2M+1}{}^{n}(z)\cos(2M+1)\pi-J_{-2M-1}{}^{n}(z)\right] \times(\sin(2M+1)\pi)^{-1},$$

which leads to a series representation in the usual Neumann form. An asymptotic series for $N_{2L+1}{}^{n}(z)$ in powers of n^{-2} for arbitrary L may be obtained. A simple form will be

$$\frac{\Gamma(n-L)n^{2L+1}}{\Gamma(n+L+1)}N_{2L+1}{}^{n}(z)\underset{|n|\to\infty}{\sim} \sum_{k=0}^{\infty} n^{-2k}\sum_{q=2k}^{3k} a_{k,q}{}^{(L)}\left(\frac{z}{2}\right)^{q} Y_{2L+1+q}(z).$$

Except when $2L+1$ equals half an odd integer, this series may be proved to be divergent but asymptotic for $|\arg(n)|\leqq\pi-\Delta<\pi$. It has the same form as (2), but with $Y_m(z)$ replacing $J_m(z)$. In Appendix IV, analogous formulas are derived for the repulsive field. [References: T. S. Kuhn, Quart. Appl. Math. **9** (1951), 1–16; Breit and Hull, Phys. Rev. **80** (1950), 392–395, 561–563; MR **14**, 45; **13**, 234, 941.]

S. C. van Veen (Delft)

ORDINARY DIFFERENTIAL EQUATIONS

See also 4175.

4027:

★Hoheisel, Guido. Aufgabensammlung zu den gewöhnlichen und partiellen Differentialgleichungen. Dritte, durchgesehene und verbesserte Aufl. Sammlung Göschen Bd. 1059. Walter de Gruyter & Co., Berlin, 1958. 124 pp. DM 2.40.

There are 3 parts, ordinary equations of first order, of higher order, partial equations, with from 5 to 10 sections in each part. Each section contains a discussion of the type of problem to be dealt with, several worked problems, then several problems to be worked, with solutions.

4028:

Beesack, Paul R. On an existence theorem for complex-valued differential equations. Amer. Math. Monthly **65** (1958), 112–115.

An alternative proof is given for a classical result concerning the existence and analyticity of solutions of equations of the form $dw/dt=f(t,w,\lambda)$, where t is real, the unknown w and the parameter λ are complex.

M. M. Peixoto (Baltimore, Md.)

4029:

Climescu, Al. C. Une contribution à la théorie des systèmes d'équations différentielles linéaires. Rev. Univ. "Al. I. Cuza" Inst. Politehn. Iaşi **1** (1954), 1–5. (Romanian. Russian and French summaries)

4030:

Wintner, Aurel. Remarks to two previous papers. Amer. J. Math. **79** (1957), 797–800.

The papers referred to are: same J. **69** (1947), 87–98; **71** (1949), 587–594; [MR **8**, 381; **11**, 33]. Let $f(t)$ be continuous on $(0,\infty)$ and consider the equations (1) $x''-f(t)x=0$, (2) $y'=y^2-f(t)$, (3) $(\log x)'=-y$, where the Riccati equation (2) is equivalent to (1) by virtue of (3) if $x(t)\neq0$. If $f(t)\geqq0$, then (1) has a solution $x(t)$ on $(0,\infty)$ satisfying $x(t)>0$, $x'(t)\leqq0$ (hence $x''(t)\geqq0$) which is unique to a positive constant factor (Kneser). The author obtains certain results about $y(t)$ in (3): (i) If $f(t)\geqq0$ on $(0,\infty)$, then (2) has a unique solution $y(t)$ satisfying $y(t)\geqq0$ on $(0,\infty)$ [restatement of Kneser's result]; (ii) if $f(t)\geqq0$ and if $f'(t)\leqq0$, then $y'(t)\leqq0$ also; (iii) if $f(t)\geqq0$, $f'(t)\leqq0$ and if, in addition, (4) $16f^3(t)\leqq27f'^2(t)$, then $y''(t)\geqq0$; (iv) if $f(t)\geqq0$ and if $f''(t)\geqq0$, then (whether or not the inequality (4) holds) the assertions of (ii) and (iii), $y'(t)\leqq0$ and $y''(t)\geqq0$, imply the assumption $f'(t)\leqq0$ of (ii). Moreover, if T denotes the class of functions $z(t)$ on $(0,\infty)$ for which $(-D)^nz(t)\geqq0$ $(n=0,1,2,\cdots)$, then the author proves: (I) $f\in T$ is necessary but not sufficient for $y\in T$ and (II) $f\in T$ is sufficient but not necessary for $x\in T$, where $x(t)$ and $y(t)$ are Kneser's solution of (1) and the corresponding function (3), respectively.

C. R. Putnam (Lafayette, Ind.)

4031:

Popov, B. S. Über die Integration der linearen Differentialgleichung dritter Ordnung in geschlossener Form. Bull. Soc. Math. Phys. Macédoine **7** (1956), 17–19. (Serbo-Croatian summary)

4032:

Bandić, I. Sur une classe d'équations différentielles quasi-homogènes du premier ordre. Boll. Un. Mat. Ital. (3) **13** (1958), 224–233.

The author treats a first order differential equation of the form (a) $u(x,y,y')=v(x,y,y')$, where u and v are homogeneous functions in the variables x and y, not necessarily of the same degree. The classical double transformation of Legendre is first applied to (a). The author then follows this with a further transformation, an appropriate differentiation, and a final substitution to obtain a new first order equation (b), in which the derivative of the unknown appears to the first degree. If a general integral of this new equation can be found, a general solution of (a) in parametric form can be written at once.

This result is utilized to form new types of first order equations which are integrable by elementary means, and to solve several of the old integrable types in a new and more systematic way.

H. L. Turrittin (Minneapolis, Minn.)

4033:

Wintner, Aurel. On a generalization of Airy's function. Arch. Rational Mech. Anal. **1** (1958), 242–245.

The normal form $D^2y=x^\mu y$ $(\mu=\mu(\lambda))$ of Bessel's equation $t^2Z''+tZ'+(t^2-\lambda)Z=0$ can be reduced in case

$\lambda=\pm\frac{1}{3}$, $\frac{2}{3}$ to (1) $D^2y-xy=0$. Moreover, putting (2) $y=A(x)=\int_0^\infty \cos(u^3-xu)du$ (Airy's function), it is seen that (3) $y=A(3^{-\frac{1}{3}}x)$ is a solution of (1). The author shows how the solution (3) of(1) can be extended so as to satisfy the equation (4) $D^{m-1}y-xy=0$ for $m\geqq 3$ (when $m=3$, then (4) reduces to (1), while if $m=2$, (4) is first order with the general solution a constant multiple of $e^{-\frac{1}{2}x^2}$). In the special case $m=3$ an integral representation of Airy's function $A(x)$ distinct from the integral (2) is also derived. In case m is even, solutions of (4) involving Cauchy's stable distributions are obtained.

C. R. Putnam (Lafayette, Ind.)

4034:

Mascart, Henri. Sur certains opérateurs linéaires différentiels. C. R. Acad. Sci. Paris **246** (1958), 3307–3309.

Let α, β be positive integers. The author studies linear differential operators having the property that $d^\alpha/dz^\alpha(L(f))=L(d^\beta/dz^\beta(f))$ for certain classes of functions. In particular, he investigates conditions on L necessary and sufficient that it shall apply to functions holomorphic in a given bounded domain and to entire functions of certain order and type. *R. E. Fullerton* (College Park, Md.)

4035:

Boetti, Giovanni. Sopra una classe di equazioni differenziali ordinarie del primo ordine dotate di una singolarità mista. I. Rend. Mat. e Appl. (5) **16** (1957), 207–220.

Consider $dy/dx=(ax+by+f(x, y))/g(x)$, $ab\neq 0$, f, g continuous in a neighborhood of the origin, $g(0)=0$, $g\neq 0$ otherwise, $\int dx/g(x)$ divergent at the right and at the left of the origin. Theorem A: If $f\equiv 0$ and g does not change signs, the characteristic behaves as in a saddle point in one of the halfplanes bounded by the y-axis, as in a node in the other halfplane; if g changes sign at $x=0$ the singularity is either a saddle point or a node. Theorem B: If f satisfies a Lipschitz condition and $|f|\leqq M|y|$ with $M<|b|$, the behavior is the same as in Theorem A.

J. L. Massera (Montevideo)

4036:

Škrášek, Josef. Hauptsystem von Lösungen einer bestimmten verallgemeinerten Eulerschen homogenen Differentialgleichung n-ter Ordnung. Acta Acad. Sci. Čechoslovenicae Basis Brunensis **27** (1955), 361–367. (Czech. Russian and German summaries)

4037:

Staržinskiĭ, V. M. Survey of articles on conditions of stability of the trivial solution of a system of linear differential equations with periodic coefficients. Advancement in Math. **4** (1958), 410–449. (Chinese)

A translation of the Russian article in Prikl. Mat. Meh. **18** (1954), 469–510 [MR **16**, 249].

4038:

Kahane, Arno. Sur certaines équations différentielles dont la solution peut être trouvée à l'aide des solutions de l'équation $\theta''=h(t)\theta$. Acad. R. P. Romîne. Stud. Cerc. Mat. **7** (1956), 307–319. (Romanian. Russian and French summaries)

L'auteur applique un théorème de Darboux [C. R. Acad. Sci. Paris **94** (1882), 1456–1459] concernant l'équation différentielle linéaire du second ordre $d^2y/dx^2=[f(x)+h]y$ (h paramètre) à un problème de mécanique et fait quelques remarques sur le théorème en question.

Voir aussi: D. S. Mitrinovitch [Bull. Acad. Sci. Math. Nat. A **6** (1939), 121–156; C. R. Acad. Sci. Paris, **208** (1939), 156–157]. *D. S. Mitrinovitch* (Belgrade)

4039:

Gamkrelidze, R. V. Theory of processes in linear systems which are optimal with respect to rapidity of action. Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 449–474. (Russian)

Continuing the work of V. G. Boltyanskiĭ, R. V. Gamkrelidze, and L. S. Pontryagin [Dokl. Akad. Nauk SSSR **110** (1956), 7–10; MR **18**, 859], the author studies the optimality problem for the solution of the linear system of differential equations,

$$x^i=f^i(x^1, \cdots, x^n; u^1, \cdots, u^r),\ i=1, \cdots, n,$$

relative to rapidity of action. The state of the physical system being considered at time t is described by the vector function $x(t)$, while $u(t)$ is a control vector. An existence proof is given, and equations for optimal direction and optimal trajectory are determined. The synthesis of the system is discussed in the case of a single direction parameter.

E. F. Beckenbach (Los Angeles, Calif.)

4040:

Wintner, Aurel. On the integration of ordinary linear differential equations by means of Fourier and related integrals. Rend. Circ. Mat. Palermo (2) **6** (1957), 289–310.

Let $q(t)$ be real-valued and continuous on $(0, \infty)$ (sometimes on $[0, \infty)$) and consider only real-valued nontrivial solutions of the differential equation (1) $x''+q(t)x=0$. As in the author's paper [Amer. J. Math. **73** (1951), 368–380; MR **13**, 37], the equation (1) is called disconjugate if no solution has more than one zero. The present paper further develops a general theory concerning "Laplace" or "Fourier" solutions of (1) which was begun by the author several years ago [ibid. **69** (1947), 87–98; **71** (1949), 587–594; MR **8**, 381; **11**, 33]. A few results are: (I) If $q(t)$ is monotone and positive and (1) is disconjugate on $[0, \infty)$, then every positive solution $x(t)$ of (1) on $[0, \infty)$ is of the form $x(t)=\exp\int_{+0}^\infty \phi(u)u^{-1}\times\sin tu du$, where $\phi(u)\geqq 0$, is continuous on $0<u<\infty$ and satisfies $\int_{+0}^\infty \phi(u)du<\infty$. (II) If $(-1)^nD^nq(t)\geqq 0$ on $(0, \infty)$ for $n=0, 1, 2, \cdots$ and if (1) is disconjugate on $(0, \infty)$, then every positive solution $x(t)$ of (1) on $(0, \infty)$ can be represented as $x(t)=\exp\int_0^\infty u^{-1}(1-e^{-tu})d\mu(u)$, where $\mu(u)$ satisfies $d\mu(u)\geqq 0$ and $\mu(0)=\mu(+0)$. (III) If $F(z)$ is an entire function of the form $F(z)=\sum_{k=0}^\infty c_kz^k$, where $c_k\geqq 0$ and $\lim_{k\to\infty} c_k^{1/k}=0$, then the differential equation $x''-F(1/t)x=0$ has on the half-plane $R(t)>0$ a solution $x(t)\not\equiv 0$ representable as a Laplace-Stieltjes transform $x(t)=\int_0^\infty e^{-tu}d_\mu(u)$ of a real non-negative distribution $\mu(u)$, $d\mu(u)\geqq 0$, and this solution is unique to a constant positive factor. In case $q(t)\leqq 0$ on $(0, \infty)$, (1) has a positive, non-increasing solution on $(0, \infty)$, unique to a constant positive factor (Kneser). Some results on Kneser's solution are obtained. In particular, it is shown that (IV) if $q(t)\leqq 0$ and $dq(t)\geqq 0$ hold on $(0, \infty)$, then the solution $r(t)=x'(t)/x(t)\leqq 0$ of Riccati's equation $r'+r^2+q(t)=0$ ($x(t)$ being Kneser's solution of (1)) satisfies $r'(t)\geqq 0$ and $r(t)\to 0$ as $t\to\infty$. *C. R. Putnam* (Lafayette, Ind.)

4041:

Hustý, Zdeněk. Asymptotische Eigenschaften der Lösungen linearer homogener Differentialgleichung der vierten Ordnung. Časopis Pěst. Mat. **83** (1958), 60–69. (Czech. Russian and German summaries)

The general linear differential equation of the fourth order is transformed into the canonical form (1) $y^{(4)}+10Ay''+(10A'+\omega)y'+[3(3A^2+A'')+\omega_1]y=0$, and some

boundedness conditions and asymptotic formulae for the solutions of (1) are given. The main device is the use of the fact that the fundamental system of the equation (1) with $\omega=\omega_1=0$ is formed by the functions u^3, u^2v, uv^2, v^3, where u and v are linearly independent solutions of the equation $u''+Au=0$. *M. Zlámal* (Brno)

4042:

Hustý, Zdeněk. Oscillatorische Eigenschaften der Lösungen einer homogenen linearen Differentialgleichung vierter Ordnung. Czechoslovak Math. J. 8(83) (1958), 62–75. (Russian. German summary)

The author studies the oscillation behaviour of the solutions of (1) [see the preceding review]. One of his results is the following theorem: Let $\omega\equiv0$ and $\omega_1\geqq0$. The solutions of (1) are oscillatory if the solutions of $y''+Ay=0$ are oscillatory. *M. Zlámal* (Brno)

4043:

Hustý, Zdeněk. Über einige Eigenschaften der homogenen linearen Differentialgleichung vierter Ordnung. Časopis Pěst. Mat. 83 (1958), 202–213. (Czech. Russian and German summaries)

A sufficient and necessary condition is given that the differential equation $y^{(4)}+4a_3y''+6a_2y'+4a_1y'+a_0y=0$ arise by the iteration of the equation $P(y)\equiv A_1y'+A_0y=0$, i.e., that it have the form $P^4(y)=0$. Further, two canonical forms of this equation are derived. *M. Zlámal* (Brno)

4044:

Greguš, Michal. Über einige neue Eigenschaften der Lösungen der Differentialgleichung $y'''+Qy'+Q'y=0$. Publ. Fac. Sci. Univ. Masaryk **1955**, 237–254. (Slovak. Russian and German summaries)

4045:

Titchmarsh, E. C. On the convergence of eigenfunction expansions. II. Quart. J. Math. Oxford Ser. (2) **8** (1957), 236–240.

Let $q(t)$ be continuous, non-decreasing, unbounded for $t\geqq0$, so that the eigenvalue problem associated with $\varphi''+(\lambda-q(t))\varphi=0$ and a boundary condition $\varphi(0)\cos\alpha+\varphi'(0)\sin\alpha=0$ has a discrete spectrum $\lambda_0<\lambda_1<\cdots$ and a corresponding complete sequence of eigenfunctions ψ_0, $\psi_1,\cdots$. Let $f(t)\sim\sum c_n\psi_n(t)$. The author had proved [same Quart. 3 (1952), 139–144; MR **14**, 50] that if $q\in C^2$, $q'>0$, $q''\geqq0$, $q''\leqq|q'|^\gamma$ for large t and $1<\gamma<4/3$ and if $f(t)$ satisfies, in a neighborhood of $t=x$, conditions to assure the convergence of ordinary Fourier series, then $f(x)=\sum c_n\psi_n(x)$ holds in the sense of ordinary convergence. In this note, it is shown that the conditions on q can be relaxed to: q is continuous, increasing, convex downwards. The proof is similar, but somewhat simpler than the old proof. It depends on the validity of an asymptotic formula for the number $N(\lambda)$ of eigenvalues $\lambda_n\leqq\lambda$ under the relaxed conditions on q [Hartman, J. London Math. Soc. 27 (1952), 492–496; MR **14**, 278]. *P. Hartman* (Baltimore, Md.)*

4046:

Opial, Z. Sur les intégrales bornées de l'équation $u''=f(t,u,u')$. Ann. Polon. Math. **4** (1958), 314–324.

First the author considers the equation (1) $u''=f(t,u)$, where $f(t,u)$ is defined and continuous in the whole plane (t,u). He proves that if there exist two numbers a, b $(a<b)$ such that $f(t,a)\leqq0$ and $f(b,t)\geqq0$ for every t, then at least one solution of (1) lies between the lines $u=a$ and $u=b$. Then the author investigates the same problem for the equation $u''=f(t,u,u')$. *M. Zlámal* (Brno)

4047:

Opial, Z. Sur l'équation différentielle $u''+a(t)u=0$. Ann. Polon. Math. **5** (1958), 77–93.

If $a(t)\geqq0$ is a continuous function of t, $0\leqq t<+\infty$, and $a(t)\to+\infty$ monotonically, then it is known that all solutions of the equation (1) $u''+a(t)u=0$ are bounded as $t\to\infty$, and there exists at least one solution which approaches zero as $t\to\infty$ [see, for example, G. Trevisan, Rend. Sem. Mat. Univ. Padova **23** (1954), 340–342; MR **16**, 589]. In addition, if $a'(t)$ is continuous and $\ln a(t)\to+\infty$ regularly as $t\to\infty$ [see G. Sansone, "Equazioni differenziali nel campo reale", vol. II, 2d ed., Zanichelli, Bologna, 1949; MR **11**, 32; p. 60], G. Armellini, L. Tonelli and G. Sansone have shown that every solution of (1) approaches zero as $t\to\infty$. The purpose of the present paper is to show that the above results hold for a more general class of functions $a(t)$. More specifically, $a(t)\geqq0$ is said to be of type (S) if $a(t)=b(t)+\psi(t)$, where $b(t)\to\infty$ monotonically as $t\to\infty$, $\psi(t)$ is of bounded variation in every finite interval and $\int_0^\infty b^{-1}(t)|d\psi(t)|<+\infty$. Also, $a(t)\geqq0$ is of type (Q) if $a(t)=\alpha(t)\beta^{-1}(t)$, where $\alpha(t)$, $\beta(t)$ are continuous, positive nondecreasing functions, $\alpha(t)\to+\infty$ as $t\to\infty$ and $\beta(t)$ is bounded for all $t\geqq0$. It is then shown that $a(t)$ is of type (S) if and only if it is of type (Q); the results of Trevisan hold if $a(t)$ is of type (Q); and the result of Armellini, Tonelli and Sansone is true if $a(t)$ is of type (S), $b'(t)$ is continuous and $\ln b(t)\to\infty$ regularly as $t\to\infty$. *J. K. Hale* (Baltimore, Md.)

4048:

Corduneanu, C. Sur l'existence des solutions bornées de systèmes d'équations différentielles non linéaires. Ann. Polon. Math. **5** (1958), 103–106.

Suppose the system of linear differential equations

$$(1)\qquad x'=A(t)x+f(t),\ x=(x_1,\cdots,x_n),\ -\infty<t<+\infty,$$

where $A(t)$ is continuous, has a k-parameter family of bounded solutions for every bounded, continuous vector $f(t)$. For every bounded solution $x(t)$ of (1) there exist positive constants, P, Q, depending only upon the matrix $A(t)$, such that

$$(2)\qquad \|x\|=\sum_{j=1}^{n}\sup_t|x_j(t)|\leqq P\|x(0)\|+Q\|f\|.$$

Under these conditions on (1) the author proves two theorems: (1) If $f(t,x)=(f_1,\cdots,f_n)$ is continuous for all t, x; $f(t,0)$ is bounded for $-\infty<t<+\infty$, and

$$(3)\qquad \|f(t,x)-f(t,\bar{x})\|\leqq L\|x-\bar{x}\|,\ L<Q^{-1},$$

then the system of nonlinear equations

$$(4)\qquad x'=A(t)x+f(t,x)$$

has a k-parameter family of solutions which are bounded for $-\infty<t<+\infty$. (2) If $f(t,x)$ is continuous for $-\infty<t<+\infty$, $\|x\|\leqq a$, $Q\|f(t,0)\|<a(1-L,Q)$, then there is a k-parameter family of bounded solutions with $\|x\|<a$. These theorems are proved by first triangularizing (1) according to S. Diliberto [Ann. Math. Studies no. 20, 1–38, Princeton Univ. Press, 1950; MR **11**, 665] and then applying the fixed point theorem of Banach [see also R. Bellman, Ann. of Math. (2) **49** (1948), 515–522; MR **10**, 121]. Also, an immediate consequence of the triangularization procedure is a necessary and sufficient condition for system (1) to satisfy the above property. *J. K. Hale* (Baltimore, Md.)

4049:

Reissig, Rolf. Methoden zur qualitativen Untersuchung nichtlinearer Schwingungen. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **7** (1957/58), 195–198. (Russian, English and French summaries)

The paper is a lecture on geometric methods for studying the qualitative behavior of nonlinear systems of one degree of freedom. After giving conditions that make possible the construction of a simple closed curve in the phase plane with the property that the phase trajectories cannot leave the region bounded by the curve, the author indicates how information concerning forced and free oscillations can be obtained. No references are given.

J. P. LaSalle (Baltimore, Md.)

4050:

Shintani, Hisayoshi. On the paths of an analytic two dimensional autonomous system in a neighborhood of an isolated critical point. J. Sci. Hiroshima Univ. Ser. A **21** (1957/58), 209–218.

Consider the system (1) $\dot{x}=ax+by+f^*(x, y)$, $\dot{y}=cx+dy+g^*(x, y)$, where f^* and g^* are $o(r)$ as $r=(x^2+y^2)^{1/2}\to 0$, $a^2+b^2+c^2+d^2\neq 0$, and both the roots of the characteristic equation are zero. When neither root is zero, the behavior of the paths of (1) near the origin is well known. When one root is zero, the behavior of the paths is also known [Bendixson, Acta Math. **24** (1901), 1–88]. The case of two zero roots has been investigated previously by Gubar [Dokl. Akad. Nauk SSSR **95** (1954), 435–438; MR **16**, 360]. Andreev [Vestnik Leningrad Univ. **10** (1955), no. 8, 43–65; MR **17**, 364–365], and Barocio [Contributions to the theory of nonlinear oscillations, vol. 3, pp. 127–135, Princeton Univ. Press, 1956; MR **19**, 145]. In this paper the author obtains, essentially, seven distinct phase portraits for the paths of (1) near the origin. He uses a theorem due to Lonn [Math. Z. **44** (1938), 507–530] on the directions of approach to a critical point and a theorem of Keil [Jber. Deutsch. Math. Verein **57** (1955), 111–132; MR **16**, 1023] relating to the number of paths tending to the origin in certain directions. Though the author's methods differ from those of Gubar, Andreev, Barocio, and Keil, his results are the same.

C. Colemam (Baltimore, Md.)

4051:

Opial, Z. Sur les intégrales oscillantes de l'équation différentielle $u''+f(t)u=0$. Ann. Polon. Math. **4** (1958), 308–313.

The author proves two theorems concerning the equation (1) $u''+f(t)u=0$, where $f(t)$ is defined and continuous for $t\geqq 0$ (in both cases he assumes the limit $\lim_{t\to\infty}\int_0^t f(\tau)d\tau$ to be final and denotes by $G(t)$ the function $\int_\infty^t f(\tau)d\tau$): 1: If $G(t)\geqq 0$ and $G(t)\not\equiv 0$ in every interval $\langle T, \infty)$ and if there exists an $\varepsilon>0$ such that, for large t, $(\frac{1}{4}+\varepsilon)G(t)\leqq\int_t^\infty [G(\tau)]^2d\tau$, then every solution of (1) is oscillatory; 2: if $G(t)\geqq 0$, and for large t, $\int_t^\infty[G(\tau)]^2d\tau=\frac{1}{4}G(t)$, every solution of (1) is non-oscillatory.

M. Zlámal (Brno)

4052:

Opial, Z. Sur un théorème de A. Filippoff. Ann. Polon. Math. **5** (1958), 67–75.

Consider the second order equation (1) $x''+f(x)x'+g(x)=0$, where $f(-x)=-f(x)$, $g(-x)=-g(x)$ are continuous for all real x. It is known that every solution of (1) is periodic for the initial values sufficiently small, provided that $g(x)$ is large enough when compared with $f(x)$. E. McHarg [J. London Math. Soc. **22** (1947), 83–85; MR **9**, 435] gave estimates for $g(x)$ in terms of $f(x)$ when both $xf(x)>0$, $xg(x)>0$. More recently, A. F. Filippoff [Mat. Sb. N.S. **30(72)** (1952), 171–180; MR **13**, 944] gave estimates for $g(x)$ when only $xg(x)>0$. The purpose of the present paper is to prove the following result: Equation (1) has every solution periodic for the initial values sufficiently small, provided there exists an $a>0$ such that

$$(2)\qquad \int_0^x \frac{g(\xi)}{|F(\xi)|}\,d\xi\geqq(\tfrac{1}{4}+\varepsilon)|F(x)|,\ 0\leqq x\leqq a,\ \varepsilon>0,$$

where $F(x)=\int_0^x f(\xi)d\xi$. If there exists an $a>0$ such that the inequality in (2) is reversed for $\varepsilon=0$, then one can always find nonperiodic solutions of (1) no matter how small the initial values. After the author gives a qualitative discussion of the behavior of the solutions in the two dimensional phase space, the proof of the above results are quite elementary. For more general statements concerning the existence (but no quantitative estimates as above) of families of periodic solutions for weakly nonlinear systems of differential equations of order >2, see the paper of J. K. Hale [Riv. Mat. Univ. Parma **5** (1954), 281–311; MR **17**, 1088; also L. Cesari and J. K. Hale, Proc. Amer. Math. Soc. **8** (1957) 757–764; MR **19**, 142]. *J. K. Hale* (Baltimore, Md.)

4053:

Borůvka, Otakar. Remarques sur le compte rendu de M. M. I. Yelchine concernant mon mémoire: "Sur les intégrales oscillatoires des équations différentielles linéaires du second ordre". Czechoslovak Math. J. **6(81)** (1956), 431–433. (Russian. French summary)

Reply to the review in RŽ Mat **1956** #406, of the article in same J. **3(78)** (1953), 199–255 [MR **15**, 706].

4054:

Sandor, Ştefan. Quelques critériums de non-oscillation. Com. Acad. R. P. Romîne **6** (1956), 753–756. (Romanian. Russian and French summaries)

Mettant à profit la définition de la non-oscillation donnée par R. L. Sternberg [Duke Math. J. **19** (1952), 311–322; MR **14**, 50], l'auteur indique quelques critériums de non-oscillation pour les systèmes de la forme

$$\frac{d}{dt}\left(A(t)\frac{dy}{dt}\right)+B(t)y=0,$$

où $B(t)$ est une matrice symétrique, A une matrice positivement définie et y un vecteur à k dimensions.

Ces critériums généralisent les critériums analogues, indiqués par A. Wintner [Amer. J. Math. **73** (1951), 368–380; MR **13**, 37] pour l'équation du second degré.

D. S. Mitrinovitch (Belgrade)

4055:

Ghizzetti, Aldo. Su una particolare equazione differenziale ordinaria non lineare. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 262–269.

Autonomous differential equations of the form, $x''+f(x')+x=0$, are considered, $f'(x')>0$ and $x'f(x')>0$ for $x'\neq 0$. Solutions of equations of this sort have the property that $x(t)$ and $x'(t)\to 0$ as $t\to\infty$. By elementary methods, separation theorems for the zeros of $x(t)$ and its derivatives are obtained, from which the author deduces criteria for distinguishing the oscillating case from the non-oscillating case. In the former case (where $x(t)$ has infinitely many zeros without being identically zero), a nomographic device is given for the rapid calculation of the solution for $t>0$. But to set up the nomogram it is necessary to integrate the equation numerically over a limited interval under several initial conditions.

D. C. Lewis, Jr. (Baltimore, Md.)

4056:
Vignoli, Paola. Sull'eccitazione parametrica nei sistemi a due gradi di libertà. Rend. Sem. Mat. Univ. Padova 27 (1957), 375–386.

The stroboscopic method is used to study solutions of the system of equations

$$L_1\ddot{q}_1+M\ddot{q}_2+\frac{q_1}{C_1}=-\frac{q_1}{C_1}\eta\cos\Omega t-\dot{q}_1g(\dot{q}_1),$$

$$M\ddot{q}_1+L_2\ddot{q}_2+\frac{q_2}{C_2}=0,$$

which govern the behavior of a certain two-mesh electrical circuit containing a periodically varying capacitance. The phenomena discussed (resonance effects, limitations of the amplitudes of oscillations by non-linearities, etc.) are mostly familiar to electronic engineers; and the paper is noteworthy chiefly because it affords a good example of the use of the method.

L. A. MacColl (New York, N.Y.)

4057:
Erugin, N. P. Qualitative methods in the theory of stability. Advancement in Math. 3 (1957), 63–84. (Chinese)

A translation of the Russian original in Prikl. Mat. Meh. 19 (1955), 599–616 [MR 17, 366].

4058:
Mikołajska, Z. Remarque sur la possibilité d'un passage continu conservant la stabilité entre deux systèmes d'équations différentielles quelconques ayant une solution périodique et stable. Ann. Polon. Math. 5 (1958), 45–53.

Suppose that $X=(x_1, \cdots, x_n)$, each $F_i(t, X)=(f_{i1}, \cdots, f_{in})$, $i=0, 1$, is continuous in t, X for $0\leqq t<+\infty$ and all X; $X=\Phi_i(t)=\Phi_i(t+T)$, $T\geqq 0$, is a periodic solution of the system (S_i) $dX/dt=F_i(t, X)$, $i=0, 1$, and this periodic solution possesses a certain property (P). The author considers the following question: Is it possible to find functions $F(t, X, \lambda)$, $\Phi(t, \lambda)$, continuous in (t, X, λ) for $0\leqq t<+\infty$, $0\leqq\lambda\leqq 1$ and all X, such that $\Phi(t, \lambda)$ is a periodic solution of period T of the equation (S_λ) $dX/dt=F(t, X, \lambda)$; $\Phi(t, \lambda)$ has the property (P) for every λ; and $F(t, X, i)=F_i(t, X)$, $\Phi(t, i)=\Phi_i(t)$, $i=0, 1$? The question is answered in the affirmative when the property (P) is the property of being either asymptotically stable or stable in the sense of Liapounoff.

J. K. Hale (Baltimore, Md.)

4059:
Zubov, V. I. On a reduction principle. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 228–230. (Russian)

Take a system (1) $\dot{y}=f(t, x, y)$, $\dot{x}=g(t, x, y)$, where x is an n-vector and y a k-vector. The system is considered in $\|x\|$, $\|y\|<H$, $t\geqq 0$ (Euclidean norms), and it is assumed that $f(t, x, 0)=0$, $g(t, 0, 0)=0$ and that f, g are continuous in the above set. Stability and asymptotic stability of the origin are defined as usual. Replace now x by an arbitrary differentiable vector $x(t)$, $\|x(t)\|<H_1$, and consider (2) $\dot{y}=f(t, x(t), y)$. If there is an H_1 such that stability or asymptotic stability hold for (2) (for $y=0$) whatever $x(t)$ as above, then $y=0$ is said to be strongly stable or asymptotically stable.

Let now $W(t, x, y)$ be such that $W(t, 0, 0)=0$, $t\geqq 0$. Let $f(r)$ be continuous, >0, except that $f(0)=0$. Then W is said to be strongly definite negative if it is definite negative for every $W(t, x, y(x))$, $\|y(x)\|\leqq f(\|x\|)$. The analogues of Lyapunov's stability theorems are proved for (1) with such a W. Other analogous stability results are given when f is a power series. [References: Malkin, Akad. Nauk SSSR Prikl. Mat. Meh. 6 (1942), 411–448; 18 (1954), 129–138, 459–463, 681–704; MR 4, 225; 15, 873; 16, 249, 590; Zubov, Dokl. Akad. Nauk SSSR 100 (1955), 857–859; Methods of A. M. Lyapunov and their application, Izdat. Leningrad Univ., Moscow, 1957; MR 16, 924; 19, 275; Kamenkov, Kazansk. Aviats. Inst. Sb. Nauč. Trudy 1939 no. 9]. *S. Lefschetz* (Mexico, D.F.)

4060:
Maravall Casesnoves, Dario. On stability of equilibrium defined for a system of differential equations of first order and its interest in biology. Gac. Mat., Madrid, 9 (1957), 166–169. (Spanish)

4061:
Vasil'eva, A. B. On repeated differentiation with respect to the parameter of solutions of simultaneous ordinary differential equations with a small parameter in the derivative term. Dokl. Akad. Nauk SSSR (N.S.) 119 (1958), 9–11. (Russian)

The system under discussion is

$$\mu\dot{z}=F(z, x, t),\quad \dot{x}=f(z, x, t),\quad z|_{t=0}=z^0,\quad x|_{t=0}=x^0,$$

where μ is a small positive parameter. A certain type of solution, tending as $\mu\to 0$ to that of

$$F(z, x, t)=0,\quad \dot{x}=f(z, x, t),$$

with the same initial values, was dealt with in previous papers and the partials with respect to μ were calculated under the assumption that f possessed the first and F the second partial with respect to μ. This is now extended to the partials of order n under the assumption that f possesses those of order n, and F those of order $n+1$. [References: Tihonov, Mat. Sb. N.S. 22(64) (1948), 193–204; 31(73) (1952), 575–586; MR 9, 588; 14, 1085; Vasilieva, ibid. 28(70) (1951), 131–146; 31(73) (1952), 587–644; Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 9 (1954), no. 3, 29–40; MR 13, 37; 14, 1086; 16, 362].

S. Lefschetz (Mexico, D.F.)

4062:
Grobov, V. A. A method for averaging standard equations containing a "quasi-cyclic" angular coordinate. Dokl. Akad. Nauk SSSR (N.S.) 119 (1958), 858–860. (Russian)

Let a dynamical system be referred to coordinates q_i, p_i, $i=1, 2, \cdots, r$ plus one angle φ and associated p_{r+1}. Let the system be represented by

$$\dot{q}_h=\frac{\partial H}{\partial p_h},\quad \dot{p}_h=-\frac{\partial H}{\partial q_h},\quad h\leqq r,$$

$$\dot{\varphi}=\frac{\partial H}{\partial p_{r+1}},\quad \dot{p}_{r+1}=-\mu\frac{\partial H_1}{\partial\varphi},$$

$$H=H_0(q_1, \cdots, q_r, p_1, \cdots, p_{r+1})$$
$$+\mu H_1(q_1, \cdots, q_r, \varphi, p_1, \cdots, p_{r+1})+\cdots,$$
$$H_0=\tfrac{1}{2}\sum a_{ik}q_iq_k+\sum b_{ik}q_ip_k+\tfrac{1}{2}\sum c_kp_k^2,$$

with μ a small parameter. The equations show that p_{r+1} is a slow varying coordinate (in the sense of Bogoliubov). On physical grounds (in the problems to which this is meant to apply), the q_h are periodic functions of period 2π in φ. An approximation

$$q_h=a_h\cos(\varphi+\psi_h)$$

is applied, where a_h, ψ_h are slowly varying variables. The treatment is then analogous to that of Krylov and Bogo-

liubov combined with an averaging of φ. [References: Bogoliubov, "On some statistical methods in mathematical physics", Akad. Nauk Ukrainsk. SSR, Kiev, 1945; MR 8, 37; Krylov and Bogoliubov, "Introduction to non-linear mechanics", Ann. of Math. Studies, no. 11, Princeton Univ. Press, 1943; MR 4, 142].

S. Lefschetz (Mexico, D.F.)

4063:

Ryabov, Yu. A. Estimations of the convergence region of periodic series representing solutions of differential equations involving a small parameter. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 642–645. (Russian)

Take the n-vector system

$$\dot{x}=Ax+\mu\varphi(t)+\mu X(x, t),$$

where A is a constant matrix, φ periodic of period 2π, X a vector whose components are power series in those of x with coefficients periodic with period 2π in t. Let r of the characteristic roots of A (all simple) be of the form mi (resonance). One may find a periodic solution of the form

$$x=\mu x^1+\mu^2x^2+\cdots.$$

This is done (more or less by brute force) and bounds are given for the coefficients. They are improvements on those found by D. C. Lewis [Duke Math. J. 22 (1955), 39–56; MR 17, 38] for the non-resonance case.

S. Lefschetz (Mexico, D.F.)

4064:

Briš, N. I. On the first boundary problem for an ordinary differential equation with a small parameter in the highest derivative. Minsk. Gos. Ped. Inst. A. M. Gor'k. Uč. Zap. 7 (1957), 3–10. (Russian)

The author considers the boundary value problem

$$(1)\quad \begin{cases} L_\varepsilon(y)\equiv\varepsilon(-1)^n(p_n(x)y^{(n)})^{(n)}+\sum_{k=0}^{n-1}(-1)^k(p_k(x)y^{(k)})^{(k)} \\ \qquad\qquad =f(x) \\ y_\varepsilon^{(k)}(a)=y_\varepsilon^{(k)}(b)=0 \quad (k=0, 1, \cdots, n-1), \end{cases}$$

where $\varepsilon>0$ is a small parameter. It is assumed that $p_n(x)\in C^n$, $p_k(x)\in C^{k+2}$ $(k=0, 1, \cdots, n-1)$ and $f(x)\in C^2$. Moreover, the signs of the $p_j(x)$ are such that the solutions of (1) and the corresponding degenerate problem

$$(2)\quad L_0(u)=f(x),\quad u^{(k)}(a)=u^{(k)}(b)=0 \quad (k=0, 1, \cdots, n-2),$$

exist and are unique. It is shown that the solution $y_\varepsilon(x)$ of (1) can be written in the form $y_\varepsilon(x)=u(x)+z(x, \varepsilon)+v(x, \varepsilon)$, where $u(x)$ is the solution of (2) and $v^{(k)}(x)=O(\varepsilon^{\frac{1}{2}})$ uniformly as $\varepsilon\to0+$ for $k=0, 1, \cdots, n-1$. $z(x, \varepsilon)$ is a boundary layer term which satisfies $z^{(k)}(x, \varepsilon)=O(\varepsilon^{\frac{1}{2}})$ for $k=0, 1, \cdots, n-2$ and

$$|z^{(n-1)}(x, \varepsilon)|\leq|u^{(n-1)}(a)|\exp\{-m^{\frac{1}{2}}(x-a)/\varepsilon^{\frac{1}{2}}\} + |u^{(n-1)}(b)|\exp\{-m^{\frac{1}{2}}(b-x)/\varepsilon^{\frac{1}{2}}\}+C\varepsilon^{\frac{1}{2}},$$

where $C>0$ is a constant and $m=\min p_{n-1}(x)/\max p_n(x)$. The proof is carried out by induction on n; the case $n=1$ being treated directly by the maximum principle. In an earlier paper [same Uč. Zap. 6 (1957)] the author has obtained the explicit form of the boundary layer term under more stringent regularity assumptions on the coefficients.

D. G. Aronson (Minneapolis, Minn.)

4065:

✱Pinney, Edmund. Ordinary difference-differential equations. University of California Press, Berkeley-Los Angeles, 1958. xii+262 pp. $5.00.

A difference-differential equation is one involving a function $y_{h_1,\cdots,h_s}(t)$, certain of the ordinary derivatives of y with respect to t, and certain of the differences of y with respect to the h_i's. Included in this general set of functional equations is the "mixed" difference-differential equation in which differences with respect to t appear, as well as derivatives with respect to t and differences with respect to the h_i's. The present work presents an extensive study of linear equations of both kinds, together with certain non-linear mixed difference-differential equations.

The first three chapters cover the general theory of linear equations. The author first surveys in chapter I a variety of methods of solution. Integral transformation techniques are the most powerful methods, employing the Laplace, Fourier, and Euler-Laplace transforms and their inverses. The method of generating functions, used when only integral values of the displacement variables are involved, is developed by replacing the transform integrals by series sums which are evaluated and re-expanded to obtain the solution. Laplace-Stieltjes integral representations, power series representations, and iterative methods are also considered. Chapter II is devoted to the development of a general theory of integro-differential equations with constant coefficients, which in most general form are represented by the system:

$$\sum_{j=1}^{l}\sum_{\mu=0}^{m}C_{ij\mu}y_j^{(\mu)}(t)+\sum_{j=1}^{l}\sum_{\mu=0}^{n}\sum_{\nu=1}^{N}D_{ij\mu\nu}y_j^{(\mu)}(t-\xi_\nu) + \sum_{j=1}^{l}\sum_{\mu=0}^{n}\int_\gamma^\delta\varphi_{ij\mu}(\xi)y_j^{(\mu)}(t-\xi)d\xi=w_i(t)\quad (i=1, \cdots, l),$$

in which the C's and D's are constants, m and n are non-negative integers, l and N are positive integers, and $0<\gamma\leq\xi_1<\xi_2<\cdots<\xi_N\leq\delta<\infty$. If the φ's are set equal to zero, there results a difference-differential system $(l>1)$ or a single equation $(l=1)$. If the w's are equal to zero, the "reduced" system, or equation, is obtained. For these equations solutions are obtained by substituting e^{zt}, in which expression z satisfies a "characteristic equation" of exponential polynomial form. Various sets of boundary conditions on y and its derivatives are listed, together with additional conditions on the φ and w functions. A number of series expansion theorems for the solutions $y_i(t)$ and their derivatives are proved. In chapter III the characteristic equations related to the reduced systems are further studied. Asymptotic expressions for the roots and numerical methods for the computation of the roots are determined. A study of the adjustment of the parameters of an equation to give the solutions certain desired properties, such as maximum damping, concludes the chapter.

The next set of four chapters deals with the solution of particular types of equations in illustration of the general theory. Chapter IV deals with the first order mixed difference-differential equation with constant coefficients and one lag:

$$y'(t)+ay'(t-1)+by(t-1)=w(t).$$

With appropriate boundary conditions this equation has a unique solution with an appropriate series representation dependent in form on the roots of the characteristic equation $ze^z+az+b=0$. Solutions are developed in series form for the cases of all simple roots, a double root, or a triple root. Chapter V treats the second order mixed difference-differential equation considered by the author to be the mixed equation most important in applications:

$$y''(t)+dy(t)+ay''(t-1)+by'(t-1)+cy(t-1)=w(t),$$

where a, b, c, d are constants. With appropriate boundary

conditions and conditions on $w(t)$, series solutions dependent on the roots of the characteristic equation $(z^2+d)e^z+az^2+bz+c=0$ are developed for various cases of root multiplicity up to and including quintuple roots. In chapter VI the author discusses six particular types of mixed difference-differential equations that have appeared in the literature. In each case boundary conditions and other restrictions are catalogued, the characteristic equation and its asymptotic roots given, and the series expansions of the solutions determined for the cases in which the characteristic roots are all simple. Chapter VII extends the discussion of particular equations begun in chapter VI to non-mixed difference-differential equations which have two independent variables, with differentiation with respect to one and differencing with respect to the other. Ten equations of this type, prominent in the literature, are solved under stated boundary conditions, usually by Euler-Laplace transform methods, with the solutions represented in series form.

Chapter VIII contains an outline of methods of reducing multiple index equations, those with more than one variable with respect to which differences are taken, to the case of single index equations like those of chapter VII. The latter half of chapter VIII discusses the reduction of a number of functional equations known in the literature to difference-differential equations and systems.

The last three chapters are devoted to the study of non-linear equations. Chapter IX presents a methods for constructing approximate solutions to the mixed difference-differential equation with small non-linear terms:

$$\sum_{\mu=0}^{m} C_\mu y^{(\mu)}(t)+\sum_{\mu=0}^{m-1}\sum_{\nu=1}^{N} D_{\mu\nu}y^{(\mu)}(t-\xi_\nu)=\varepsilon f(Y(t), t),$$

wherein ε and the C's, D's, and ξ's are independent of t, and f is a polynomial of degree n in the elements of the matrix $Y(t)=(y^{(\mu)}(t-\varepsilon_\nu))$ and a polynomial in circular functions of t. The study of the general theory of the equation is here subordinate to the determination of approximate solutions with a view toward applications in the mathematical theory of economic processes. In chapter X the methods developed in the previous chapter are applied to five typical examples of ordinary differential equations and mixed difference-differential equations. Chapter XI studies Minorsky's equation:

$$y''(t)+2ry'(t)+\omega^2y(t)+2qy'(t-1)=\varepsilon y'^3(t-1),$$

where r, q, ω, ε are constant and ε is small. The associated characteristic equation, $z^2+2rz+\omega^2+2qze^{-z}=0$, and the asymptotic forms of its roots are studied. The theory of chapter IX is applied to the construction of solutions in the non-critical case, in which the characteristic roots having greatest real part are all simple, and in the critical case in which the characteristic roots are double.

In the appendix existence and uniqueness theorems are established for a system of integro-differential equations sufficiently general to include all cases considered in the book. A bibliography of nearly two hundred titles gives reference to related mathematical investigations and applications in many fields ranging from dynamical systems, vibrations and oscillations, and fluid flow to nuclear physics and the theory of business cycles.

The author has performed most commendably an extensive task in assembling and arranging the inevitably large mass of detail involved in studying functional equations of this kind and developing explicit solutions in sufficient detail to render them useful in applications to problems in other fields.

P. E. Guenther (Cleveland, Ohio)

4066:

✱Kolmogorov, A. N. Théorie générale des systèmes dynamiques et mécanique classique. Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 315–333. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 852 pp. $7.00.

The author presents a survey of the field of dynamical and canonical systems of differential equations, i.e., systems

$$\frac{dx_\alpha}{dt}=F_\alpha(x_1, \cdots, x_n) \quad (\alpha=1, \cdots, n)$$

which possess an integral invariant $\int M(x)dx_1\cdots dx_n$ and systems of the form

$$\frac{dq_\alpha}{dt}=\frac{\partial H}{\partial p_\alpha}, \quad \frac{dp_\alpha}{dt}=-\frac{\partial M}{\partial p_\alpha} \quad (\alpha=1, \cdots, s).$$

The introduction contains an interesting discussion of the diverse methods which penetrated this field during its development, such as the methods of topology measure theory, spectral theory, theory of continuous groups, etc. It is explained that it is reasonable, in the case of these systems, to ask for a description of the majority of solutions, possibly neglecting sets of measure zero (in contradistinction to the study of limit cycles, etc. in the theory of regulating systems and problems which involve damping). Moreover, it is the goal of such a qualitative theory to characterize a type of systems which is not exceptional and represents the "general case." In particular, the following concept is introduced: A certain property of a system is called stable if it is preserved under small perturbations of the system.

In section 2 the program of such a qualitative theory is carried out for the systems of differential equations on a two dimensional torus. It contains a description of Kolmogorov's result [Dokl. Akad. Nauk SSSR **93** (1953), 763–766; MR **16**, 37] which states that such an analytic system takes the form $dx/dt=\lambda_1$, $dy/dt=\lambda_2$ (λ_1, λ_2, constants) in appropriate coordinates, provided the rotation number $\lambda_1/\lambda_2=\gamma$ satisfies $|\gamma-p/q|\geq ch^q$ for all integers p, q and some constants c, $h>0$.

In section 3 almost periodic solutions of canonical systems are discussed and their existence is claimed to be a stable property in the sense mentioned above. (For more details see below.)

Section 4 contains a discussion of flows on noncompact manifolds, E. Hopf's results on surfaces with negative curvature and the capture problem for the 3-body problem. The results of J. Chazy, G. A. Merman, O. Y. Schmidt and K. A. Sitnikov are mentioned.

The spectral theory of flows is discussed in Section 5. It contains Grabar's example of an analytic irreducible flow which is not strongly ergodic. Furthermore, the interesting papers of Gel'fand and Fomin are stated in which methods of group representations are used to determine the spectrum of the geodesic flow on a compact surface with negative curvature.

In the center of this talk is the author's new statement on the conservation of conditionally periodic solutions: Consider a canonical system with the real analytic Hamiltonian $H(q, p, \theta)=W(p)+\theta S(q, p)$, where θ is a small parameter. The variables $q=(q_1, \cdots, q_s)$ are angular variables of period, say 2π. For $\theta=0$ this system is integrable and the phase space can be decomposed into invariant tori $p_\alpha=c_\alpha$, on which the flow is given by $q_\alpha=q_\alpha(0)+tW_{p_\alpha}(c)$. Consider a torus $T(0)$ for which the

rotation numbers $W_{p_\alpha}(c)=\lambda_\alpha$ satisfy the inequality $|\sum n_\alpha\lambda_\alpha|\geq c(\sum|n_\alpha|)^{-\eta}$ for all integers n_α and some c, $\eta>0$, and assume $\det(W_{p_\alpha p_\beta})\neq 0$. Then it is claimed that, for sufficiently small θ, there exist analytic tori $T(\theta)$ which are invariant under the perturbed flow. The rotation numbers on $T(\theta)$ are the same as on $T(0)$, namely λ_α. (The proof of this theorem was published in Dokl. Akad. Nauk SSSR **98** (1954), 527–530 [MR **16**, 924], but the convergence discussion does not seem convincing to the reviewer.) This very interesting theorem would imply that for an analytic canonical system which is close to an integrable one, all solutions but a set of small measure lie on invariant tori. *J. Moser* (Cambridge, Mass.)

4067:

Avdeev, N. Ya. Hyperbolic solution of ordinary differential equations of first order. Rostov. Gos. Ped. Inst. Uč. Zap. **4** (1957), 61–68. (Russian)

This note considers the conditions under which an ordinary differential equation of first order has for its solution $u(x, y)=c$ a function $u=u(x, y)$ satisfying the equation of the vibrating string. *From the introduction*

4068:

Avdeev, N. Ya. Harmonic solution of a Pfaff equation. Rostov. Gos. Ped. Inst. Uč. Zap. **4** (1957), 75–78. (Russian)

4069:

Mlak, W. A note on non-local existence of solutions of ordinary differential equations. Ann. Polon. Math. **4** (1958), 344–347.

Let E be a Banach space and the function $f(x,t)$ be defined on the Cartesian product $E\times\langle t_0, t_0+\beta\rangle$ with values lying in E. The author proves two theorems about the non-local existence of the solution of the differential equation $x'=f(x,t)$ with the initial condition $x(t_0)=x_0$. The assumptions are too long to be stated here. The method used is that of Leray-Schauder. *M. Zlámal* (Brno)

PARTIAL DIFFERENTIAL EQUATIONS

See also 4027, 4105, 4297, 4353, 4380, 4438.

4070:

Andreev, A. F.; and Bogdanov, Yu. S. On continuous dependence of the solution of a Cauchy problem on the initial data. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 3 (81), 165–166. (Russian)

The authors introduce a metric into the space of solutions of the Cauchy problem such that in some cases uniqueness of the solution for each of a large class of initial conditions is related to the continuous dependence of the solution on its initial values.

M. M. Day (Urbana, Ill.)

4071:

Duff, G. F. D. Mixed problems for linear systems of first order equations. Canad. J. Math. **10** (1958), 127–160.

For a system of linear partial differential equations of the form

$$(1)\quad a_{rs}{}^i\frac{\partial u_s}{\partial x^i}+b_{rs}u_s=f_r\quad (r, s=1, \cdots, R, i=1, \cdots, N)$$

the author studies auxiliary data problems in which conditions are prescribed on two distinct surfaces having an intersection of lower dimension. The existence of one or more real characteristic surfaces is assumed, but the system (1) need not be hyperbolic. Denote by G: $\varphi(x^i)=0$ a real characteristic surface of multiplicity μ, and by T: $\psi(x^i)=0$ a non-characteristic surface which intersects G in an edge C of dimension $N-2$. Prescribe conditions on G of the form $e_r{}^\lambda u_r=g^\lambda$, $\lambda=1, \cdots, R-\mu$, and data on T of the form $a_r{}^\nu u_r=g^\nu$, $\nu=R-\mu+1, \cdots, R$, where the $e_r{}^\lambda$, $a_r{}^\nu$ are subject to suitable restrictions. The author proves that if the data and coefficients are analytic and if a certain functional determinant Δ does not vanish, then there exists a unique analytic solution of (1) in a quadrant bounded by G and T which satisfies the prescribed conditions. A sufficient condition that $\Delta\neq 0$ is that $\mu=1$. The solution is based on the method of dominant power series, and is made to depend on the solution in G of a related partial differential equation, with initial data given on C. This result is applied by the author to study a mixed problem for a quadrant bounded by two non-characteristic surfaces S and T, which contains at least $k_0\geq 1$ characteristic surfaces G_i, of multiplicity $\mu=1$, which pass through $C=S\cap T$. It is shown that there exists a solution which takes Cauchy data on S, is continuous across G_i, $i=1, \cdots, k_0$, and which satisfies suitable boundary conditions on T. Under additional conditions the solution is shown to be unique in the class of all $C^{(1)}$ solutions assuming the same data, and in the case that (1) is symmetric hyperbolic the existence is demonstrated also for non-analytic coefficients and data.

The author shows that in the case of characteristic surfaces of multiplicity $\mu>1$ the solution of the related equation in G depends on data over the entire surface T, not just the edge C, and, hence, is not directly applicable to a mixed problem involving two non-characteristic surfaces. *R. Finn* (Berlin)

4072:

Olaru, V. Le problème de Goursat-Beudon pour l'équation linéaire aux dérivées partielles d'ordre n à deux variables indépendantes. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 191–208. (Romanian. Russian and French summaries)

An equation of the form

$$\frac{\partial^n u}{\partial x^k\partial y^{n-k}}=b_1\frac{\partial^n u}{\partial x^{k-1}\partial y^{n-k+1}}+b_2\frac{\partial^n u}{\partial x^{k-2}\partial y^{n-k+2}}+\cdots+b_k\frac{\partial^n u}{\partial y^n}+\cdots$$

has solutions with given values on the characteristics; there are at most $(n-1)$ distinct solutions; on one characteristic, one to $(n-1)$ can be given, and on the other curve (which may also be a characteristic), from $(n-1)$ to one must be given.

An equation of the degenerate form

$$\frac{\partial^{n-h}u}{\partial x^p\partial y^{n-p-h}}=a_n\frac{\partial^n u}{\partial y^n}+a_{n-1}\frac{\partial^{n-1}u}{\partial y^{n-1}}+\cdots+a_{n-h+1}\frac{\partial^{n-h+1}u}{\partial y^{n-h+1}}+c_1\frac{\partial^{n-h}u}{\partial x^{p-1}\partial y^{n-h-p+1}}+c_2\frac{\partial^{n-h}u}{\partial x^{p-2}\partial y^{n-h-p+2}}+\cdots+c_p\frac{\partial^{n-h}u}{\partial y^{n-h}}+\cdots$$

also has solutions with given values on the characteristics; in this case there are at most $(n-h)$ distinct

solutions; on one characteristic one to $(n-h)$ may be given, and on the other curve (which may also be a characteristic) from $(n-h-1)$ to zero must be given. The existence proof takes a, b, c as analytic functions of the variables x and y. *D. J. Struik* (Cambridge, Mass.)

4073:

✱Synge, J. L. The hypercircle in mathematical physics: a method for the approximate solution of boundary value problems. Cambridge University Press, New York, 1957. xii+424 pp. $13.50.

This book is concerned with the description of a method of approximate solution applicable to a wide class of boundary value problems. Consider a real linear vector space with positive semi-definite scalar product. It is desired to find upper and lower bounds for the square of the length $(v, v)=|v|^2$ of an a priori unknown vector v, which may be, for instance, the solution, or the gradient of the solution, of a given boundary value problem. However, some information is known about v, say the boundary conditions and the differential equation satisfied by v (compare the example below), so that it is possible to find known vectors y and z such that $(y-v, z-v)=0$, that is, such that

$$|v-\tfrac{1}{2}(y+z)|^2=|\tfrac{1}{2}(y-z)|^2,$$

which means, in geometric language, that the unknown vector v lies on the "hypersphere" S with center $\frac{1}{2}(y+z)$ and radius $\frac{1}{2}|y-z|$.

From this, using the triangle inequality, we get the desired bounds

$$(\tfrac{1}{2}|y+z|-\tfrac{1}{2}|y-z|)^2 \leqq |v|^2 \leqq (\tfrac{1}{2}|y+z|+\tfrac{1}{2}|y-z|)^2,$$

as may be visualized at once in Euclidean 3-space.

More generally, it is often possible, using only the information available about v, to find, in addition to the above vectors y and z, an orthonormal set of $(p+q)$ vectors $y_1, y_2, \cdots, y_p, z_1, z_2, \cdots, z_q$ such that $(y_i, z-v)=(y-v, z_j)=0$. Then it follows, as before, that v lies on the intersection of the above hypersphere S with the "hyperplane" P of vectors u satisfying the $p+q$ linear equations $(u, y_i)=(z, y_i)$ and $(u, z_j)=(y, z_j)$. It is natural to call this intersection a "hypercircle". Using the triangle inequality in P, we get numerically computable upper and lower bounds on $|v|^2-\sum (z, y_i)^2-\sum (y, z_j)^2$ generalizing those above on $|v|^2$. To give an application, consider the problem of estimating the Dirichlet integral $\int_D (u_\xi^2+u_\eta^2)d\xi d\eta$ for the solution $u(\xi, \eta)$ of the Poisson equation $u_{\xi\xi}+u_{\eta\eta}=F(\xi, \eta)$ in a bounded plane domain D having a smooth boundary C with $u=f$ on C. The above linear vector space may then be chosen to consist of all ordered pairs $[p_1, p_2]$ of sufficiently smooth real-valued functions defined on $D+C$, the scalar product being given by $([p_1, p_2], [q_1, q_2])=\int_D (p_1q_1+p_2q_2)d\xi d\eta$. The unknown vector v is $[u_\xi, u_\eta]$ and we seek upper and lower bounds for $(v, v)=|v|^2=\int_D (u_\xi^2+u_\eta^2)d\xi d\eta$. The essential fact used in the selection of the various vectors y_i, z_j mentioned above is the following orthogonality relation: if $[p_1, p_2]=[w_\xi, w_\eta]$, that is, if $[p_1, p_2]$ is the piecewise continuous gradient of a continuous function $w(\xi, \eta)$, which vanishes on C, and if the piecewise continuous functions q_1 and q_2 satisfy $\partial q_1/\partial\xi+\partial q_2/\partial\eta=0$ in D and $[q_1, q_2]$ has continuous normal component $q_1n_1+q_2n_2$ across any curve in D, then $([p_1, p_2], [q_1, q_2])=0$, by Green's theorem. The vector $z=[r_\xi, r_\eta]$ is the gradient of a sufficiently smooth function $r(\xi, \eta)$ for which $r=f$ on the boundary C, while $z_j=[\partial r_j/\partial\xi, \partial r_j/\partial\eta]$ with $r_j=0$ on C. Also $y=[y_1, y_2]$ with $\partial y_1/\partial\xi+\partial y_2/\partial\eta=F$ in D, while $y_i=[y_{i1}, y_{i2}]$, with $\partial y_{i1}/\partial\xi+\partial y_{i2}/\partial\eta=0$ in D. Upper and lower bounds for the actual values, at a given point, of the unknown solution v (or of its derivatives) of a boundary value problem can be dealt with in a similar way.

We now give a brief account of the general outline of the book. There are three main parts, entitled: No metric, Positive-definite metric, and Indefinite metric; corresponding, respectively, to (real) linear vector spaces, linear vector spaces with a positive definite scalar product, and linear vector spaces with a scalar product not necessarily positive definite. Part I (=chapter 1) is mainly of an introductory nature. Part II is concerned with geometrical considerations in a linear vector space with a positive definite scalar product. Chapter 2 ends with a section entitled "The key to the hypercircle method", and this is where the present review started. Chapter 3 has the title "The Dirichlet problem for a finite domain in the Euclidean plane" and chapter 4 "The torsion problem". Chapter 5 deals with various boundary value problems, for example, the equilibrium of an elastic body. Part III contains two chapters, one on geometry and the other one on vibration problems. Somewhat loosely phrased, the general idea is that the minimum principles of part II become variational principles in part III.

The printing and format of the book are excellent. The exposition is of the highest order; many an exquisitely turned phrase is to be found among these pages. There is a wealth of figures and every section ends with a set of exercises for the reader. The author's keen concern for actual numerical results is evident from the many specific examples which he has worked out in detail, using a hand computer. Of special interest in this regard is his clear distinction, at the end of chapter 2, between reliable and unreliable bounds, relative to the usual methods of numerical computation to a certain number of significant figures. The author's point about "the practical computer (who claims to have solved a set of equations, when he has not, strictly speaking)" is very well taken. *J. B. Diaz* (College Park, Md.)

4074:

Nehari, Zeev. On the principal frequency of a membrane. Pacific J. Math. 8 (1958), 285–293.

A set of three theorems related to Rayleigh's theorem that the homogeneous circular membrane has the lowest principal frequency among all homogeneous membranes of the same mass; namely: Theorem I. If λ is the principal frequency of a membrane of given mass whose density distribution $p(x, y)$ is such that $\log p(x, y)$ is subharmonic, then $\lambda\geqq\lambda_0$, where λ_0 is the principal frequency of a homogeneous circular membrane of the same mass.

Theorem II. If λ is the principal frequency of a circular membrane of given mass whose density distribution $p(x, y)$ is superharmonic, then $\lambda\leqq\lambda_0$, where λ_0 is the principal frequency of a homogeneous circular membrane of the same mass.

Theorem III. Let α be an analytic subarc of C which is concave with respect to D. If Λ denotes the principal frequency of a homogeneous membrane whose boundary is free along α and fixed along $C-\alpha$, then $\Lambda\geqq\Lambda_0$, where Λ_0 is the principal frequency of a homogeneous semi-circular membrane of equal mass whose boundary is free along the diameter and fixed along the semi-circle.

S. H. Gould (Providence, R.I.)

4075:

de Castro, Antonio. On a mixed boundary problem. Gac. Mat., Madrid, 10 (1958), 6–12. (Spanish)

The purpose of this article is to give a numerical method of approximating the solution $u(x, y)$ of $u_{xx}+u_{yy}=0$ in the interior of the rectangle $0\leqq x\leqq a$, $0\leqq y\leqq 2b$, with the boundary conditions $u(x, 0)=f_0(x)$, $u(x, 2b)=f(x)$; $u_x|_{x=0}=u_x|_{x=a}=0$ for $0\leqq y\leqq b$; $u(0, y)=u(a, y)=g(y)$ for $b\leqq y\leqq 2b$.
From the introduction

4076:

Krivenkov, Yu. P. On a representation of the solutions of the Euler-Poisson-Darboux equation. Dokl. Akad. Nauk SSSR (N.S.) 116 (1957), 351–354. (Russian)

According to Henrici [Comment. Math. Helv. 27 (1953), 235–293; MR 15, 710], the analytic solutions w of the equation

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{c}{y}\frac{\partial w}{\partial y}=0 \quad (c=\text{const}>0)$$

on a convex region symmetrical about the x-axis can be represented in terms of analytic functions φ as

$$w(x, y)=\frac{\Gamma(c)}{\Gamma^2(c/2)}\int_0^1 \frac{\varphi[x+iy(1-2\sigma)]}{[\sigma(1-\sigma)]^{1-c/2}}\,d\sigma.$$

This result is extended here to functions $w\in C''$ on a region T bounded by a segment L of the x-axis and such that $T\cup L\cup \bar{T}$ (where $\bar{T}$ is the image of T about the x-axis) is convex. Again c is taken positive, but for $0<c<1$ there is adjoined the further condition $\lim_{y\to 0} y^c(\partial w/\partial y)=0$.
M. G. Arsove (Seattle, Wash.)

4077:

Krivenkov, Yu. P. A representation of solutions of Euler-Poisson-Darboux's equation by analytic functions. Dokl. Akad. Nauk SSSR (N.S.) 116 (1957), 545–548. (Russian)

As an extension of his previous paper [#4076 above] the author proves the following. Let T be a region of the complex plane which has an interval L of the x-axis as part of its boundary. If a solution $w(x, y)$ of

$$(*)\qquad \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{c}{y}\frac{\partial w}{\partial y}=0$$

in T, where c is a real constant such that $0<c<1$, has continuous second derivatives in T or if $\lim_{y\to 0} y^c(\partial w/\partial y)$ is an analytic function of x on L, then there is a region σ such that L is part of the boundary of σ and such that, in σ, solution $w(x, y)$ may be written:

$$(**)\quad w(x, y)=\gamma\left(\frac{c}{2}\right)\int_0^1 \frac{\varphi[x+iy(1-2\sigma)]}{[\sigma(1-\sigma)]^{1-c/2}}\,d\sigma +\gamma\left(1-\frac{c}{2}\right)\left(\frac{y}{1-c}\right)^{1-c}\int_0^1 \frac{\psi[x+iy(1-2\sigma)]}{[\sigma(1-\sigma)]^{c/2}}\,d\sigma,$$

where $\gamma(c/2)=\Gamma(c)/\Gamma^2(c/2)$ and functions $\varphi(z)$ and $\psi(z)$ are analytic in $\sigma\cup L\cup\bar{\sigma}$ (where $\bar{\sigma}=[\bar{z}|z\in\sigma]$); and on L, $\varphi(x)=w(x, 0)$ and $\psi(x)=\lim_{y\to 0}(y/(1-c))^c\partial w/\partial y$. Definition: Functions $w(x, y)$ and $w^*(x, y)$, both with domain T, are said to be conjugate if each has continuous second derivatives in T and the following conditions are satisfied on L: $w(x, 0)=w^*(x, 0)$; and $\lim_{y\to 0} y^c(\partial w/\partial y)=-\lim_{y\to 0} y^c(\partial w^*/\partial y)$. Each of the following is a necessary and sufficient condition that a solution $w(x, y)$ of (*) in T with continuous second derivatives in T can be represented in the form (**) in T: (i) there is a solution $w^*(x, y)$ of (*) in T which is conjugate to $w(x, y)$; (ii) there exists a function $\varphi(z)$, analytic in T, such that on L, $\varphi(x)=w(x, 0)$; (iii) there exists a function $\psi(z)$, analytic in T, such that on L, $\psi(x)=\lim_{y\to 0}(y/(1-c))^c(\partial w/\partial y)$. Finally, let $D=T\cup L\cup\bar{T}$. An arbitrary solution $w(x, y)$ of (*) in D, which has continuous second derivatives in D, may be written in the form

$$w(x, y)=\gamma\left(\frac{c}{2}\right)\int_0^1 \frac{\varphi[x+iy(1-2\sigma)]d\sigma}{[\sigma(1-\sigma)]^{1-c/2}} +\operatorname{sign} y\cdot\gamma\left(1-\frac{c}{2}\right)\left(\frac{|y|}{1-c}\right)^{1-c}\int_0^1 \frac{\psi[x+iy(1-2\sigma)]d\sigma}{[\sigma(1-\sigma)]^{c/2}}$$

where $\varphi(z)$ and $\psi(z)$ are functions analytic in D and satisfying on L the conditions: $\varphi(x)=w(x, 0)$ and $\psi(x)=\lim_{y\to 0}(|y|/(1-c))^c(\partial w/\partial y)$.
J. Cronin (Elizabeth, N.J.)

4078:

Pilat, B. Sur les extrêmes des fonctions composées par des intégrales des équations aux dérivées partielles du type elliptique. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 6 (1958), 317–319.

A result due to M. Biernacki [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 24 (1958), 7–11; MR 20 #3387] concerning solutions of elliptic partial differential inequalities of second order containing no mixed derivatives is generalized by removing the last mentioned assumption.
A. Huber (Muenchenstein)

4079:

Protter, M. H. A maximum principle for hyperbolic equations in a neighborhood of an initial line. Trans. Amer. Math. Soc. 87 (1958), 119–129.

The reviewer has proved [Ann. of Math. (2) 64 (1956), 505–513; MR 19, 1058] that if u is a solution of the differential equation

$$(*)\quad Lu\equiv(au_x)_x-(bu_y)_y+cu_x+du_y+fu=0,\ a>0,\ b>0,$$

such that $u\leqq 0$ and $\partial u/\partial y\leqq 0$ at $y=0$, and if the coefficients of L satisfy certain inequalities in a characteristic triangle for $y>0$, the solution attains its maximum on the initial line. The same conclusion holds a fortiori in a union of such characteristic triangles, and in particular for the intersection of a characteristic triangle with a strip $0\leqq y\leqq y_0$.

Introducing the function $v=e^{\gamma x}(1-\beta e^{-\alpha y})$, the author shows that for any equation (*) and any characteristic triangle T there exist constants $y_0>0$, $\gamma\geqq 0$, $\alpha\geqq 0$, $0\leqq\beta<1$ such that if $u\leqq 0$ and $\partial(u/v)/\partial y\leqq 0$ at $y=0$ and if $Lu\geqq 0$, then the maximum of u in the intersection of T and the strip $0\leqq y\leqq y_0$ occurs at $y=0$.

Various applications to ordinary differential equations are made.
H. F. Weinberger (College Park, Md.)

4080:

Copson, E. T. On a singular boundary value problem for an equation of hyperbolic type. Arch. Rational Mech. Anal. 1 (1958), 349–356.

The author is concerned with a solution of the partial differential equation

$$U_{xx}+2\alpha U_x/x=U_{yy}+2\beta U_y/y,$$

where α and β are positive constants, such that:

(i) U and its first derivatives are continuous in $x\geqq 0$, $y\geqq 0$;

(ii) the second derivatives of U are continuous in $x>0$, $y>0$;

(iii) $U=f(x)$ when $y=0$, $x\geqq 0$; and $U=g(y)$ when $x=0$, $y\geqq 0$; $f(0)=g(0)$.

The fact that the data are given on singular lines of the differential equation upsets the usual arguments and, as things stand, the problem has no solution unless f and g are suitably related. *A. E. Heins* (Ann Arbor, Mich.)

4081:

Datzeff, Assène. Sur le problème de Stefan (problème de congélation) au cas de deux ou trois dimensions. Ann. Univ. Sofia Fac. Sci. Phys. Math. Livre 2. 48 (1953/54), 33–76 (1954). (Bulgarian summary)

The problem of Stefan refers to a class of problems in which the solution of a partial differential equation (or system) is sought which satisfies given conditions on the boundaries, which are themselves unknown and are sought in the problem. As a typical example of such a problem, one might consider the problem of finding the temperature distribution in the melting of a piece of ice which is completely submerged in water. The water and ice represent the two states, with the surface of the ice being an unknown boundary which is also to be found as part of the solution of the problem.

Let A_1 and A_2 be two homogeneous bodies which represent the solid and liquid phases, for example water (A_1) and ice (A_2). A_1 and A_2 are infinitely long cylindrical bodies which are in contact along a cylindrical surface. Select an xy-coordinate system, normal to the surface generators, which intersects the two bodies in two plane domains $R_1(A_1)$ and $R_2(A_2)$ which are separated by a curve l_0. Let the curve l_0 be a simple closed regular curve enclosing the bounded interior domain R_2. The exterior domain R_1 is infinite and doubly connected. Let $u^1(x, y, t)$ and $u^2(x, y, t)$ be the temperatures in A_1 and A_2 respectively, ρ_i, σ_i, k_i $(i=1, 2)$ their densities, specific heats and conductivities, and δ the heat of fusion. We assume that $\rho_1=\rho_2=\rho$. Then u^1 and u^2 satisfy the heat equations

$$(1)\qquad a_i^2\Delta u^i=\frac{\partial u^i}{\partial t}\quad\left(\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2};\ i=1, 2\right)$$

and the initial conditions (2) $u^i(x, y, t_0)=\Phi_i(x, y)$, where Φ_i are bounded integrable functions in their respective domains.

In a rectangular system of coordinate $Oxyt$, a variable contour l and the initial contour $l_0=l(t_0)$ bound a surface L on which the functions u^i should satisfy the conditions (3) $u^i[L]=0$. The intersections of L with the planes $t=$ constant are the "level lines" l_t such that $u^i=0$. Let D be a one-parameter family on L of orthogonal trajectories of l_t and let d be the family of curves which are the orthogonal projections of D onto the xy-plane. If $s(\alpha, t)$ is the arc length of a curve d measured from the curve l_t, then the d family has the parametric representations (4) $x=f(\alpha, s(\alpha, t))$, $y=g(\alpha, s(\alpha, t))$, where α is a parameter which varies from curve to curve. The functions f, g and s are connected through the relations (5) $(\partial f/\partial t)^2+(\partial g/\partial t)^2=(\partial s/\partial t)^2$, (6) $(\partial f/\partial t)(\partial f/\partial\alpha)+(\partial g/\partial t)(\partial g/\partial\alpha)=0$. On the surface L the condition of Stefan must hold, that is,

$$(7)\qquad \frac{\partial s}{\partial t}=e\left(k_1\frac{\partial u^1}{\partial \nu}-k_2\frac{\partial u^2}{\partial \nu}\right)_{\substack{x=f(\alpha,s(\alpha,t))\\ y=g(\alpha,s(\alpha,t))}},\quad e=\frac{1}{\rho\delta},$$

where ν is the interior normal to l_t.

Now the problem of Stefan can be formulated as follows: find the functions u^1, u^2, f, g, and s satisfying the conditions (1), (2), (3), (5), (6) and (7).

An auxiliary problem is first considered. In the region R bounded by L and the planes $t=t_0$ and $t=t_0+T$ is sought a function $u(x, y, t)$ satisfying the equation (8) $a^2\Delta u=\partial u/\partial t$ and the conditions (9) $u(x, y, t_0)=\Phi(x, y)$ and (10) $u(x, y, t)=0$ on L $(t_0<t\leq t_0+T)$.

The following procedure is used to solve this problem. The interval (t_0, t_0+T) is divided into n parts at the arbitrary times $t_0, t_1, \cdots, t_n=t_0+T$. At each instant t_i is constructed a plane perpendicular to the t-axis which cuts the surface L in the curve l_i. A right cylinder with base bounded by l_i is constructed with generators parallel to the t-axis, the lateral surface being designated by L_{in}. In the enclosed region R_i, the function $u_{i,n}$ is the solution of (8) which satisfies the conditions $u_{i,n}(x, y, t_{i-1})=u_{i-1,n}(x, y, t_{i-1})$ and $u_{1n}(x, y, t_0)=\Phi(x, y)$. This leads to the sequence of functions (11) $u_{in}=V_{in}+W_{in}$, where

$$(12)\quad V_{in}=\frac{1}{4\pi a^2(t-t_{i-1})}\times\iint_{R_{i-1}}u_{i-1,n}(\xi, \eta, t_{i-1})\exp\left\{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2(t-t_{i-1})}\right\}d\xi d\eta,$$

$$(13)\quad W_{in}=\frac{1}{4\pi a^2}\int_{t_{i-1}}^{t}d\tau\int_{l_{i-1}}\frac{\mu(\sigma, \tau-t_{i-1})}{(t-\tau)^2}\times\exp\left\{-\frac{r^2}{4a^2(t-\tau)}\right\}r\cos(r\nu)d\sigma.$$

Here r is the distance from an arbitrary point $M_0(x_0, y_0)$ on l_{i-1} to a point $M(x, y)$ interior to l_{i-1}, ν is the interior normal to l_{i-1}, and $\mu(\sigma, t)$ is a solution of

$$(14)\quad \mu(\sigma, t)-\frac{1}{4\pi a^2}\int_{t_{i-1}}^{t}d\tau\int_{l_{i-1}}\frac{\mu(\sigma', \tau-t_{i-1})}{(t-\tau)^2}\times\exp\left\{-\frac{r^2}{4a(t-\tau)}\right\}r\cos(r\nu)d\sigma'=V_{in}(\sigma, t).$$

The solution of this integral equation is found in series form in each region R_i, which in turn leads to a sequence u_{in} which is shown to converge to a function $u(x, y, t)$ satisfying (8), (9) and (10).

This method is used to find solutions $u_{in}^{(1)}$, $u_{in}^{(2)}$ of equation (1), which by virtue of (7) permits the computation

$$(15)\quad s_{in}(\alpha, t)-s_{i-1,n}(\alpha, t_{i-1})=e\int_{t_{i-1}}^{t}\left(k_1\frac{\partial u_{in}^{(1)}}{\partial \nu_i}-k_2\frac{\partial u_{in}^{(2)}}{\partial \nu_i}\right)dt.$$

This in turn leads to the sequential determination of $f_{i-1,n}$ and $g_{i-1,n}$, so that $u_{in}^{(1)}$, $u_{in}^{(2)}$ and s_{in}, f_{in}, g_{in} are all determined. Taking the limit as $n\to\infty$ leads to the solution of the problem.

The generalization of the preceding to the case of three dimensions is discussed in detail.

C. G. Maple (Ames, Iowa)

4082:

Zeragiya, P. K. Boundary problems for certain non-linear equations of parabolic type. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 24 (1957), 195–221. (Russian)

The author considers the first and second boundary problems in a space-time cylinder, with sufficiently smooth lateral surface, for the non-linear parabolic equation

$$(1)\qquad \Delta u-\frac{\partial u}{\partial t}=f(x, t, u),$$

where Δ is the n-dimensional Laplace operator. It is assumed that f and $f_u>0$ are continuous and that f satisfies a Lipschitz condition with respect to $x=(x_1, \cdots, x_n)$. The existence of unique solutions of these problems is proved by constructing convergent monotone

sequences of upper and lower functions. An upper [lower] function for (1) is a function which satisfies the boundary conditions under consideration and the differential inequality $\Delta v-\partial v/\partial t-f(x, t, v)<0$ [$\geqq 0$]. The proof depends on the following comparison theorem, which is an extension of a result due to S. A. Chaplygin for elliptic equations [A new method of approximate integration of differential equations, Gosudarstv. Izvest. Tehn.-Teor. Lit., Moscow-Leningrad, 1950]. If u is a solution of the given boundary problem for(1) and v is an upper [lower] function for (1), then $v\geqq u$ [$v\leqq u$] in the whole cylinder considered. The first boundary problem for $\partial^2u/\partial x^2-\partial u/\partial t=f(x, t, u, \partial u/\partial x)$ is also treated by the same method. Here $f(x, t, u, p)$ must satisfy, among other conditions, $f_{px}-2f_u<0$ and $f_{uu}\alpha^2+2f_{up}\alpha\beta+f_{pp}\beta^2<0$ for all real α, β such that $\alpha^2+\beta^2\neq 0$. The paper also includes an existence theorem for the Cauchy problem for a fourth order non-linear parabolic equation which is proved by successive approximations.

D. G. Aronson (Minneapolis, Minn.)

4083:

Zeragiya, P. K. Application of the Čaplygin method to obtaining an approximate solution of a non-linear equation of parabolic type. Soobšč. Akad. Nauk Gruzin. SSR **18** (1957), no. 6, 647–654. (Russian)

The method of Chaplygin employed in the paper reviewed above is applied to the first boundary problem for

$$(1)\qquad \sum_{i,j=1}^{3} a_{ij}(x, t)\frac{\partial^2 u}{\partial x_i\partial x_j}=f\left(x, t, u, \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial t}\right),$$

with $\sum a_{ij}\xi_i\xi_j$ positive definite and $f(x, t, u, p, q)$ satisfying $f_u>0$, $f_q>0$ and $(\xi_0(\partial/\partial u)+\sum_{i=1}^{3}\xi_i(\partial/\partial p_i)+\xi_4(\partial/\partial q))^2f\leqq 0$. An algorithm is given for constructing monotone sequences of upper and lower functions for (1) which converge to the solution of the boundary problem. Thus, if there exists at least one upper and lower function for (1), the unique solution of the boundary problem exists. Unlike the paper reviewed above, the question of the existence of upper and lower functions is not considered.

D. G. Aronson (Minneapolis, Minn.)

4084:

Krzyżański, Miroslaw. Recherches concernant l'allure des solutions de l'équation du type parabolique lorsque la variable du temps tend vers l'infini. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 23 (1957), 28–32.

This paper contains several results analogous to some previously given by the author [Bull. Acad. Polon. Sci. Cl. III 4 (1956), 247–251; MR 18, 47]. Consider the equation $L[u]\equiv a(x, t)u_{xx}''+b(x, t)u_x'+c(x, t)u-u_t'=0$, where $(x, t)\in Q:\{x\geqq 0,\ t\geqq 0\}$; here, $a>0$ and b are continuous and bounded and $c<0$ is continuous, in the interior of Q. Under hypothesis (A), given $k>0$ and functions $\varphi(x)$ (continuous, $0\leqq x\leqq k$), $\psi_0(t)$ and $\psi_1(t)$ (each continuous, $t\geqq 0$) such that $\varphi(0)=\psi_0(0)$, $\varphi(k)=\psi_1(0)$, there exists a solution $u(x, t)$, regular in the half-strip $0\leqq x\leqq k$, $t\geqq 0$, and satisfying on the boundary the conditions $u(x, 0)=\varphi(x)$, $u(0, t)=\psi_0(t)$, $u(k, t)=\psi_1(t)$. (In some cases the author apparently includes the case in which the half-strip is replaced by Q.) A function $F(x, t)$ continuous in Q is of class E_α if there exist constants $M>0$, $K>0$ such that $|F(x, t)|<M\exp K|x|^\alpha$ when $(x, t)\in Q$.

A number of theorems are stated, without proof. Among these is the following: Assume that there exists a regular function $V(x, t)$ of class E_2 in Q such that (1) $V(x, t)>0$ in Q and $V(x,0)$ has a positive lower bound for $x\geqq 0$; (2) $L[V]\leqq 0$ in the interior of Q and $L[V]$ is of class E_2; and (3) $\lim_{t\to\infty}V(x, t)=0$, uniformly for $x\geqq 0$; suppose also that hypothesis (A) is satisfied and that the solution $u(x, t)$ is regular and of class E_2 in Q and such that $\varphi(x)=u(x, 0)$ is bounded for $x\geqq 0$, and that $\lim_{t\to\infty}u(0, t)=0$. Then $\lim_{t\to\infty}u(x, t)=0$, uniformly with respect to x for $x\geqq 0$. The author determines functions $V(x, t)$ which satisfy the hypotheses of the theorem with reference to certain specific equations $L[u]=0$.

F. W. Perkins (Hanover, N.H.)

4085:

Pogorzelski, W. Problème aux limites aux dérivées tangentielles pour l'équation parabolique. Ann. Sci. École Norm. Sup. (3) **75** (1958), 19–35.

The object of this paper is to prove the existence of a solution of

$$(*)\qquad \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}\frac{\partial^2 u}{\partial x_\alpha\partial x_\beta}+\sum_{\alpha=1}^{n} b_\alpha\frac{\partial u}{\partial x_\alpha}+cu-\frac{\partial u}{\partial t} =F\left(x_1, \cdots, x_n, t, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right).$$

The coefficients are defined in $\Omega+S$, $0\leqq t\leqq T$, where Ω is a bounded domain in n-dimensional Euclidean space, bounded by a closed surface S. The solution sought has to satisfy the following condition: (1) it is a solution of (*) in Ω, $0<t<T$; (2) for $0<t<T$, it satisfies a limiting condition on the boundary S of the form

$$\frac{du}{dT_P}+g(P, t)u(P, t)=G[P, t, u, u_1, u_2, \cdots, u_q],$$

where d/dT_P denotes the transversal derivative at a point P of S and $u_1, u_2, \cdots, u_q$ are certain tangential derivatives at P; (3) $u\to 0$ as $t\to 0$ at every interior point of Ω.

It is shown that, under conditions too lengthy to explain in a review, this problem has at least one solution. The proof depends on Schauder's fixed-point theorem. {Note. The notation has been slightly simplified in this review.}

E. T. Copson (St. Andrews)

4086:

Haimovici, M. Sur le prolongement des équations du II-e ordre à une fonction inconnue de deux variables indépendantes et sur les transformations de ces équations. III. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) 3 (1957), 53–75. (Romanian. Russian and French summaries)

4087:

Haimovici, Adolf. Una generalizzazione del metodo di Fourier per la risoluzione di alcuni problemi ai limiti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 573–579.

This paper demonstrates the existence of an infinite family of solutions to the boundary value problem

$$u_{xx}+cu_{tt}+a(x)u_x+b(t)u_t=0,$$

$$u_x(0, t)=\alpha(t)V[u(0, t)],\quad u_x(1, t)=\beta(t)V[u(1, t)],$$

where c is a constant, b is continuous, a, α, β are twice-differentiable, and V is continuous with bounded derivative and integrable reciprocal over a given permissible range of variation of u. These solutions may be specialized to satisfy initial conditions (two if $c\neq 0$, one if $c=0$). It is proved that, under certain restrictions, the solution to the initial-boundary value problem is unique if $c\geqq 0$.

A prominent feature of the method is the extension of

the notions of eigenvalue and eigenfunction to certain nonlinear ordinary differential equation problems.

H. C. Kranzer (New York, N.Y.)

4088:

Copson, E. T. On the Riemann-Green function. Arch. Rational Mech. Anal. **1** (1958), 324–348.

This paper surveys the known methods of finding the Riemann-Green function and lists all of the major known cases. The use which is being made of this function in some recent developments of partial differential equations makes this account very useful, since the author has provided the reader with the evolutionary ideas as well as specific examples. *A. E. Heins* (Ann Arbor, Mich.)

4089:

Pachale, Helmut. Über eine Klasse nichtlinearer biharmonischer Randwertprobleme. Arch. Math. **8** (1957), 376–380.

This paper states the existence and uniqueness of the solution to certain nonlinear boundary value problems for the biharmonic equation $u=0$ in three dimensions. The boundary conditions prescribe u at one fixed boundary point p_0 and, in addition, a relation expressing the values of u and its normal derivative u_n at a general boundary point p in terms of the totality of boundary values of u, u_n, and u_{nn}. The proof involves reduction to a system of integral functional equations. Details are reserved for later publication.

H. C. Kranzer (New York, N.Y.)

4090:

Kliot-Dašinskiĭ, M. I. On the rate of convergence of the method of orthogonal projections for the first boundary problem in the case of an equation of polyharmonic type. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 566–569. (Russian)

4091:

Walter, Wolfgang. Zur Existenz ganzer Lösungen der Differentialgleichung $\Delta^p u=e^u$. Arch. Math. **9** (1958), 308–312.

It has been shown for two dimensions by H. Wittich [Math. Z. **49** (1944), 579–582; MR **6**, 228], and for more dimensions by the author [Jber. Deutsch. Math. Verein. **57** (1955), 94–102; MR **16**, 929], that the equation $\Delta u=e^u$ has no entire solutions. The author also showed [Math. Z. **67** (1957), 32–37; MR **19**, 150] that the equation $\Delta^p u=e^u$, where p is any positive integer, has no entire solutions in two dimensions.

This paper proves the surprising result that such entire solutions do exist in three or more dimensions for any $p\geqq 2$.

More general equations of the form $\Delta^p u=f(u)$ are also considered. *H. F. Weinberger* (College Park, Md.)

4092:

Morev, I. A. Solution of certain linear differential systems by means of hypercomplex monogenic functions. Ukrain. Mat. Ž. **10** (1958), no. 1, 59–69. (Russian. English summary)

For the system of partial differential equations

$$\frac{\partial w_0}{\partial x}+i\frac{\partial w_0}{\partial y}=0,\quad \frac{\partial w_1}{\partial x}+i\frac{\partial w_1}{\partial y}=-\lambda_1\frac{\partial w_0}{\partial x}-\mu_1\frac{\partial w_0}{\partial y},\ \cdots$$

$$\frac{\partial w_{r-1}}{\partial x}+i\frac{\partial w_{r-1}}{\partial y}=-\sum_{k=1}^{r-1}\lambda_k\frac{\partial w_{r-k-1}}{\partial x}-\sum_{k=1}^{r-1}\mu_k\frac{\partial w_{r-k-1}}{\partial y},$$

generalizing the Cauchy-Riemann equations, the author obtains a complete solution in terms of monogenic hypercomplex numbers. The solution is given as a hypercomplex power series representing a monogenic hypercomplex function. *E. F. Beckenbach* (Los Angeles, Calif.)

4093:

Hartman, Philip. Hölder continuity and non-linear elliptic partial differential equations. Duke Math. J. **25** (1957), 57–65.

The author considers mappings $p=p(x,y)$, $q=q(x,y)$ defined in a domain D and satisfying an inequality of the form

$$(1)\qquad p_x^2+p_y^2+q_x^2+q_y^2\leqq\sigma\partial(q,p)/\partial(x,y)+\tau$$

for suitable positive constants σ, τ, with $\sigma>2$. Let $z_0\in D$, denote by $d(z_0)$ the distance from z_0 to the boundary of D and by I the sum of the Dirichlet integrals of (p,q) extended over D. The author proves that if $|z-z_0|\leqq\frac{1}{2}d(z_0)$, then the mapping functions satisfy the Hölder condition

$$(2)\quad |p(z_0)-p(z)|+|q(z_0)-q(z)|\leqq C(I^{\frac12}+\tau^{\frac12})|z_0-z|^\lambda/d^\lambda(z_0),$$

where C depends only on σ and λ is defined by $\sigma=\lambda+\lambda^{-1}$. Since it is known that I can be estimated in compact subsets D' of D, depending only on D', σ, τ and a bound for $|p|$, (2) leads to a corresponding estimate in terms of this bound. Such a result had already been obtained by Nirenberg [Comm. Pure Appl. Math. **6** (1953), 103–156, 395; MR **16**, 367] but with a smaller value for the Hölder exponent. The result of Nirenberg does not permit the following application, given by the author. Let $\varphi(x,y)$ be a $C^{(2)}$ solution in D of $G(x,y,r,s,t)=0$, $r=\varphi_{xx}$, $s=\varphi_{xy}$, $t=\varphi_{yy}$, where G is suitably smooth as a function of its arguments, and $4G_rG_t-G_s^2>0$. Then r, s, t are Hölder continuous at all points of D. This theorem extends known results on the Hölder continuity of solutions of elliptic equations and, in some instances, provides a new and simpler proof of this property. Unfortunately, the method does not seem to yield directly the uniformity of the Hölder constant which is essential for the application to existence theorems and to a priori estimation of the solution.

A similar result on the Hölder continuity of mappings which satisfy (1) has been obtained, independently, by the reviewer and J. Serrin [#4094 below]. An application in the theory of compressible fluid flow has been given by D. Gilbarg and the reviewer [Trans. Amer. Math. Soc. **88** (1958), 375–379; MR **20** #2158]. *R. Finn* (Berlin)

4094:

Finn, Robert; and Serrin, James. On the Hölder continuity of quasi-conformal and elliptic mappings. Trans. Amer. Math. Soc. **89** (1958), 1–15.

The authors deal with continuously differentiable functions u, v of (x,y) satisfying, in a domain A, inequalities (1) $|\nabla u|^2+|\nabla v|^2\leqq 2K\partial(u,v)/\partial(x,y)+K_1$, $2K=\mu+\mu^{-1}$, $0<\mu<1$, $K_1>0$ and (2) $u^2+v^2\leqq 1$. It is shown that, on any compact subset B of A, $w=u+iv$ satisfies (3) $|w(z_1)-w(z_2)|\leqq H|z_1-z_2|^\mu$, where H depends only on μ, K_1, $d=\text{dist}(B,A)$. This result has also been obtained by the reviewer [#4093 above] and is a refinement of a result of Nirenberg [Comm. Pure Appl. Math. **6** (1953), 103–156, 395; MR **16**, 367]. The methods are similar to Morrey's estimates for the growth of the Dirichlet integral of w when $K_1=0$ [Trans. Amer. Math. Soc. **43** (1938), 126–166]. The authors obtain similar results for cases where u, v have isolated singularities and/or K_1 in (1) is

replaced by $K_1|z-z_0|^{-2\lambda}$, $0\leq\lambda<1$. They also give proof, for some known results in the case $K_1=0$. For examples in this case, H in (3) can be replaced by H/d^{μ} where H ($\leq\pi e$) is an absolute constant [Hersch and Pfluger, C. R. Acad. Sci. Paris **234** (1952), 43–45; MR **13**, 736]; also, if $z\to w$ is a 1-1 map of $|z|<1$ onto $|w|<1$, then H can be chosen to be an absolute constant [cf. Akira Mori, J. Math. Soc. Japan **8** (1956), 156–166; MR **18**, 27].

P. Hartman (Baltimore, Md.)

4095:

Yosida, Setuzô. Hukuhara's problem for hyperbolic equations with two independent variables. I. Semi-linear case. Proc. Japan Acad. **34** (1958), 319–324.

By Hukuhara's problem for the real semi-linear hyperbolic system

$$\frac{\partial u_i}{\partial t}-\lambda_i(t,x)\frac{\partial u_i}{\partial x}=f_i(t,x,u_1,u_2,\cdots,u_N),$$

where i takes the values $1, 2, 3, \cdots, N$, is meant the solution of the system, given the value of each function u_i on a corresponding curve C_i. This reduces to the Cauchy problem when all the curves C_i are identical. It is shown that, under suitable conditions, the Hukuhara problem is well-posed in Hadamard's sense.

E. T. Copson (St. Andrews)

4096:

Lax, Peter D. Asymptotic solutions of oscillatory initial value problems. Duke Math. J. **24** (1957), 627–646.

In this paper the author gives an elegant and constructive theory on initial value problems of linear hyperbolic systems (1) $Mu\equiv u_t+\sum_1^m A_j(x,t)u_{x^j}+B(x,t)u=0$ for a vector function $u=(u^1,u^2,\cdots,u^n)$, where the coefficient matrices A_j and B are sufficiently smooth functions of the variables t and $x=(x^1,x^2,\cdots,x^m)$. The matrix $\sum_1^m p_jA_j$ is assumed to have n distinct real eigenvalues at each point (x,t) for any real $(p_j)\neq0$, and hence the hyperplane $t=$const. is space-like. § 1 gives the construction of a formal asymptotic series, to the Nth term,

$$u\sim u_N=e^{i\xi l(x,t)}\sum_0^N\frac{w_k(x,t)}{\xi^k}\tag{2}$$

(l is a scalar, w_k a vector), with the initial condition (3) $u(x,0)=\phi(x)\equiv e^{i\xi l_0(x)}\psi(x)$. Let $R_j(x)$ be the complete set of right eigenvectors of $\sum_1^m l_{0x^j}A_j(x,0)$; then one may set $\psi(x)=\sum_1^n\sigma_j(x)R_j(x)$. The function $l(x,t)$ and the terms w_k in the series can be determined successively by solving initial value problems of ordinary differential equations along bicharacteristics, and in the final step one has $Mu_N=e^{i\xi l(x,t)}\cdot\xi^{-N}(Mw_N)$. The number of terms one can get by this process depends on the number of continuous derivatives possessed by A_j and B. It is shown in § 3 that the solution of the initial value problem with (1), (3) can be represented by $u(P)=\int G(x,P)\phi(x)dx$, where the influence function G is a generalized function with a finite minus l-norm; the question of dependence domain is discussed. The method of § 3 is based on the energy inequality in a way similar to that in a previous paper by the author on symmetric hyperbolic systems [Comm. Pure Appl. Math. **8** (1955), 615–633; MR **17**, 1212]. The asymptotic series is then used in § 4 to obtain "approximations" G_N of the influence function G. Let a point P be fixed, $P(x=0,\ t=T)$, and let $G(x,P)=G(x)$ and $G_N(x,P)=G_N(x)$. Consider a solution u of (1) (3), and a function $v(x,t;\xi,\omega)$ with parameters $0<\xi<\infty$, $\omega=(\omega_1,\omega_2,\cdots,\omega_m)$, $|\omega|=1$, such that, in the slab $0\leq t\leq T$, $-\infty<x<\infty$, the equation (4) $M^*v=0$ holds for the adjoint operator M^*, with the condition on the top of the slab given as follows: (5) $v(x,T;\xi,\omega)=Ie^{i\xi(x\cdot\omega)}$ ($I=$unit matrix). Green's formula in the slab gives $\int u(x,T)e^{i\xi(x\cdot\omega)}dx=\int\phi(x)v(x,0;\xi,\omega)dx$. By inverting the Fourier integral one has $u(0,T)=\int\phi(x)G(x)dx$, with (6) $G(x)=\lim_{\alpha\to0}\int_0^\infty\int_{|\omega|=1}v(x,0;\xi,\omega)e^{-\alpha\xi}\xi^{m-1}d\xi d\omega$. The approximations $G_N(x)$ are obtained by replacing, in (6), the function v by v_N and by integrating ξ from 1 (instead of 0) to ∞. Here v_N is determined from (4), (5) by solving ordinary differential equations along backward bicharacteristics in the same way as u_N was determined in § 1 by (1), (3). The following remarkable theorems are proved: $G(x)-G_N(x)$ is a smooth function; and $G_N(x)$ is smooth when x does not lie on a bicharacteristic issuing from $P(0,T)$ (generalized Huygens principle). The asymptotic series was also used in § 2 to discuss the incorrectness of initial value problems on a not-space-like hyperplane.

Yu Why Chen (Detroit, Mich.)

4097:

Bogdan-Teodorescu, Gabriela. Sur une équation aux dérivées partielles du 4e ordre. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 181–190. (Romanian. Russian and French summaries)

Dans cette Note, l'auteur résout le problème de Cauchy pour l'équation du 4e ordre,

$$\left(\frac{\partial^2}{\partial x^2}-\alpha_1\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial x^2}-\alpha_2\frac{\partial}{\partial t}\right)u=0$$

dans la bande $x\in]-\infty,+\infty[$; $t\in[0,T]$, $T>0$; avec les conditions initiales

$$u(x,0)=f_1(x),\quad\left(\frac{\partial u}{\partial t}\right)_{t=0}=f_2''(x).$$

A cet effet, on utilise la transformation de Fourier dans $R=]-\infty,+\infty[$.

Résumé de l'auteur

4098:

Korobeĭnik, Yu. F. On solution of operator equations by the method of Fourier. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. **1957**, no. 5, 71–86. (Russian)

On considère dans un espace hilbertien H l'équation $du/dt+S_1u+S_2(t)u=f(t)$, $t\in[0,l]$, $u(0)=\varphi_0$, où S_1 est un opérateur auto-adjoint >0, d'inverse complètement continu, et où les $S_2(t)$ sont non bornés dans H mais opérateurs linéaires continus de V dans H si V désigne le domaine de $S_1^{\frac{1}{2}}$ racine carrée positive de S_1 (et dépendant convenablement de t) (V est muni de la norme du graphe). Etude du problème par la méthode de Galerkine, mais avec la particularité suivante: dans la méthode de Galerkine on prend une base quelconque de V; ici — comme c'est bien naturel — on prend la base des fonctions propres de S_1. Solutions généralisées, solutions usuelles.

Il n'y a pas de résultats nouveaux; l'exposé est clair.

J. L. Lions (Nancy)

DIFFERENTIAL ALGEBRA

4099:

Seidenberg, A. Some basic theorems in partial differential algebra. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 1–8.

The author reexamines certain known theorems. In the first part he shows that the usual definition of "(differentially) algebraic" is equivalent to one using induction on the number of derivation operators. Certain desired

properties follow more easily from the first definition, and others from the second. By including all these properties and the equivalence in one inductive proof, he effects a certain economy. In the subsequent parts he proves that, in a separable differential field extension, every differential transcendence basis is separating, a result previously proved by him in the case of ordinary differential fields; and he also discusses the connection between the condition that every finitely generated extension of a differential field F be simply generated and the condition that 0 be the only differential polynomial over F vanishing identically on F. *E. R. Kolchin* (New York, N.Y.)

POTENTIAL THEORY

See also 3977, 4089.

4100:

Itô, Jun-iti. Properties of subharmonic functions in the half-plane. Duke Math. J. **25** (1958), 499–504.

The Ahlfors-Heins theorem [Ann. of Math. (2) **50** (1949), 341–346; MR **10**, 522; see also Hayman, J. Math. Pures Appl. (9) **35** (1956), 115–126; MR **17**, 1073] deals with the asymptotic behavior of a subharmonic function in a half plane when its growth is restricted along the boundary. Here, the author weakens the restrictions imposed along the boundary. Let $u(z)$ be subharmonic in $x\geqq 0$ (including ∞ but excluding 0) and say that $u(z)/x$ tends effectively to α if $\lim_{|z|\to 0} u(z)/x=\alpha$ except for a set of rays of outer logarithmic capacity 0, uniformly in any closed interior angle if $|z|$ is excluded from a set of finite logarithmic length, and without exception in an interior angle in which u is harmonic. Effective boundedness means the same thing, with convergence replaced by boundedness. The author establishes the following theorems. If (a) $\liminf_{r\to 0} r\int_{-\pi/2}^{\pi/2} |u(re^{i\phi})|\cos\phi\, d\phi=0$ and (b) $\limsup_{z\to 0} u(z)/x=\beta>-\infty$ ($|\arg z|<\delta<\pi/2$), and (c) $\int_{\to 0}^{1} t^{-2}u^+(\pm it)dt$ or (d) $\int_{\to 0}^{1} t^{-2}u(\pm it)dt$ exists, then $u(z)/x$ tends effectively to β; convergence can be replaced by boundedness in the conclusion if the same is done in (d); also, under the other hypotheses, (c) implies (d). Furthermore, (c) can be replaced by (f) $u(\pm it)\leqq At$, together with (g) $\int_{\to 0}^{1} t^{-2}\{u(it)+u(-it)\}dt$ exists and (h) $\int_r^{2r} t^{-1}u(\pm it)dt\leqq o(r)$; and replacing convergence by boundedness in (g) and dropping (h) replaces convergence by boundedness in the conclusion. Also, if $\sigma(r)=\max|u(z)|$ for $|z|=r$, then, under (a) and (b) plus (d) or (f), (g), (h), we have that $\lim_{r\to 0}\sigma(r)/r$ exists and is not negative. Finally, under (a) and (b), $\int_\delta^1 t^{-1}u(\pm it)dt=O(1)$ ($\delta\to 0$) implies that $u(z)$ is effectively bounded. *R. P. Boas, Jr.* (Evanston, Ill.)

4101:

Bugrov, Ya. S. On imbedding theorems. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 531–534. (Russian)

It is shown that if φ is a function in L_p ($1\leqq p\leqq\infty$) on the boundary of the unit disc σ and u is the harmonic function on σ having φ as boundary function (such a function u always exists), then u belongs to the Nikol'skiĭ class $H_p^{(1/p)}(\sigma)$. This appears as an analogue of the following general embedding theorem, stated without proof: if $\varphi\in L_p(R_m)$, where R_m is Euclidean m-space and $1\leqq p\leqq\infty$, then for $n\geqq m$ there is a function f in $H_{pn}^{((n-m)/p)}$ such that $f(x_1,\cdots,x_m,0,\cdots,0)=\varphi(x_1,\cdots,x_m)$. Moreover, f has the representation

$$f(x_1,\cdots,x_n)=(2\pi)^{-m}\int_{R_m(t)} K(t_1,\cdots,t_m,x_{m+1},\cdots,x_n)\times\varphi(t_1+x_1,\cdots,t_m+x_m)dt_1\cdots dt_m$$

with

$$K(t_1,\cdots,t_m,x_{m+1},\cdots,x_n)=\int_{R_m(u)}\exp[-\sum_{k=1}^m(|u_k|+1)(\sum_{j=m+1}^n x_j^2)^{\frac{1}{2}}]\exp[i\sum_{k=1}^m u_kt_k]du_1\cdots du_m.$$

A corresponding embedding theorem was proved by Nikol'skiĭ [Mat. Sb. N.S. 33(75) (1953), 261–326; MR 16, 453] for φ in $H_{pm}^{(r-(n-m)/p)}$ and f in $H_{pn}^{(r)}$, under the assumption $r-(n-m)/p>0$. The present theorem can be regarded as an extension of Nikol'skiĭ's theorem to the case of $r-(n-m)/p=0$, the convention being made that $H_{pn}^{(0)}=L_p(R_n)$. (For the Dirichlet problem analogue mentioned initially, $m=1$, $n=2$, and $(n-m)/p=1/p$.) *M. G. Arsove* (Seattle, Wash.)

4102:

Tan, Wie-hang. Use of harmonic integral kernel to solve Dirichlet problem in two dimensions. Advancement in Math. 3 (1957), 396–403. (Chinese. English summary)

Several theorems relating kernel functions and harmonic functions are proved, and some applications of them to concrete problems, such as problems of infinite strip, circular ring, ellipse, elliptic ring, region bounded by two or three arcs, and so on, are discussed. *Author's summary*

4103:

Tumarkin, G. C.; and Havinson, S. Ya. Conditions for the representability of a harmonic function by Green's formula in a multiply-connected region. Mat. Sb. N.S. **44(86)** (1958), 225–234. (Russian)

The author investigates the necessary and sufficient condition on u such that, in a multiply-connected region G with boundary Γ, $u=\int_\Gamma u(\zeta)d\omega(\zeta,z)$. The following are typical conditions which the author deduces: (i) if $\alpha(t)$ maps the universal covering surface of G onto $|t|<1$, then $u(\alpha(t))$ can be represented by its Poisson-Stieltjes integral; (ii) there exist u_1, $u_2\geqq 0$, $u=u_1-u_2$, such that u_i is the limit of a non-decreasing sequence of bounded non-negative harmonic functions; (iii) the minimal harmonic majorant of $\max(|u|-M, 0)$ tends to zero in G as $M\to\infty$; (iv) the set of integrals $\int_E u(z)d\omega^i(z)$ is uniformly absolutely continuous with respect to ω^i, where E is taken on the Jordan boundary curves of $G^i\subseteq G$, such that $\lim G^i=G$ and $\omega^i(z)=\omega^i(E,z,G^i)$. *František Wolf* (Berkeley, Calif.)

4104:

Nicolescu, Miron. Le problème de l'analyticité des fonctions réelles. Rev. Math. Pures Appl. 2 (1957), 53–59.

The author notes that there are problems of analyticity of real functions that do not have their source in the theory of analytic functions of one or several complex variables. To make his point, he discusses hyperbolic, elliptic, and parabolic analyticity.

Let

$$\Delta_2 f(x,y)=f(x+h,y+k)-f(x+h,y)+f(x,y)-f(x,y+k).$$

The function F is said to be hyperbolically continuous at (x,y) provided $\lim_{h,k\to 0}\Delta_2 f/(hk)$ exists and is finite; then this limit is said to be the hyperbolic derivative of f at (x,y). The hyperbolic-derivative analogue of the Taylor series of a function is studied.

Similar considerations of analogous elliptic and parabolic differential operators lead to the study of polyharmonic and harmonically analytic functions on the one hand, and of polycaloric and parabolically analytic functions on the other.

E. F. Beckenbach (Los Angeles, Calif.)

4105:

Nicolescu, M. Problème de l'analyticité par rapport à un opérateur linéaire. Studia Math. **16** (1958), 353–363.

Soit $\mathscr{B}$ un anneau commutatif avec unité e; soit A un endomorphisme surjectif de $\mathscr{B}$ et soit t tel que $At=e$. On pose $Bx=Atx-tAx-x$. On suppose vérifiés les axiomes suivants: (i) $A^m x=0$ entraîne $A^m Bx=0$; (ii) si $A^m x=0$, il existe $y\in\mathscr{B}$ avec $A^{m-1}y=A(x-ty)=0$; ces axiomes sont vérifiés en particulier si $B=0$, auquel cas A est appelé opérateur parabolique simple, par analogie avec l'opérateur $\partial^2/\partial u^2-\partial/\partial v$ défini sur l'algèbre des fonctions indéfiniment dérivables des deux variables u et v.

On montre que toute solution de l'équation itérée $A^m x=0$ peut s'écrire $x=x_0+tx_1+\cdots+t^{m-1}x_{m-1}$, avec $Ax_i=0$ ($i=0, 1, \cdots, m-1$). Ceci généralise le théorème d'Almansi relatif au développement des fonctions harmoniques d'ordre m.

On considère ensuite le cas où $\mathscr{B}$ est une algèbre de Banach et on se pose le problème suivant: tout élément $x\in\mathscr{B}$ avec $\|A^m x\|$ borné ($m=0, 1, \cdots$) peut-il être mis sous la forme $x=\sum_{m=0}^{\infty} x_m t^m$, avec $Ax_m=0$? On donne une réponse affirmative lorsque certains axiomes supplémentaires sont vérifies, et des applications à l'opérateur parabolique et au Laplacien. {Il serait intéressant de rapprocher cette dernière application de la théorie des fonctions harmoniques d'ordre infini de N. Aronszajn [Acta Math. **65** (1935), 1–156].}

J. Deny (Strasbourg)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 4351, 4352, 4353.

4106:

Young, G. S. The linear functional equation. Amer. Math. Monthly **65** (1958), 37–38.

A short but elegant proof, suitable for presenting to elementary classes, of the fact that any solution of the functional equation $f(x+y)=f(x)+f(y)$ which is bounded on an interval is of the form mx.

J. W. Green (Los Angeles, Calif.)

4107:

Bellman, Richard; and Lehman, Sherman. Functional equations in the theory of dynamic programming. IX. Variational analysis, analytic continuation, and imbedding of operators. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 905–907.

Announcement of results to appear in full in the future.

4108:

Gossu, M. [Hosszú, M.] Unsymmetric means. Colloq. Math. **5** (1957), 32–42. (Russian)

Continuous unsymmetric quasi-linear interior means, or continuous functions of the form

$$m(x, y)=\phi[pf(x)+qf(y)],\quad p>0,\quad q>0,\quad p+q=1,$$

with $\min(x, y)<m(x, y)<\max(x, y)$ for $x\neq y$, where f is a strictly monotonic function and ϕ is its inverse, have been given several characterizations in the class of continuous functions $x\cdot y$ of two real variables. Thus J. Aczel [Bull. Amer. Math. Soc. **54** (1948), 392–400; MR **9**, 501] characterized them by the conditions: (1) $(x\cdot y)\cdot(u\cdot z)=(x\cdot u)\cdot(y\cdot z)$; (2) $x\cdot x=x$; (3) $x\cdot u>y\cdot u$ and $u\cdot x>u\cdot y$ for $x>y$. L. Fuchs [Acta Math. Acad. Sci. Hungar. **1** (1950), 303–320; MR **13**, 922] replaced (3) by the weaker condition (4) $x\cdot u\neq y\cdot u$ and $u\cdot x\neq u\cdot y$ for $x\neq y$, giving a characterization in terms of (1), (2), and (4). In the class of functions satisfying (5) $x\cdot y$ is twice continuously differentiable, J. Aczel then [Colloq. Math. **4** (1956), 33–55; MR **18**, 876] replaced (1) and (2) by (6) $(x\cdot y)\cdot z=(x\cdot z)\cdot(y\cdot z)$, giving a characterization — in this class of functions — in terms of (4), (5), and (6). The present author showed [Publ. Math. Debrecen **3** (1953), 83–86; MR **15**, 962], after the preceding result had been communicated to him orally, that in the class of continuous functions the condition (5) can be dropped if the condition (7) $z\cdot(x\cdot y)=(z\cdot x)\cdot(z\cdot y)$ is added; he thus gave a characterization in terms of (4), (6), and (7). In the present paper he shows that condition (4) can be dropped, yielding a characterization in terms of conditions (6) and (7).

E. F. Beckenbach (Los Angeles, Calif.)

4109:

Hosszú, Miklós. Non-symmetric means. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7** (1957), 207–218. (Hungarian)

Dieser Artikel enthält die Arbeit des Verf. [#4108] sowie weitere Bemerkungen, darunter ein explizites Beispiel einer kürzbaren (regulären) Struktur, die mit einer nur von einer Seite autodistributiven binären Operation versehen ist. Da diese Operation von der anderen Seite nicht autodistributiv ist, ist sie auch nicht bisymmetrisch.

J. Aczél (Debrecen)

4110:

Bajraktarević, Mahmud. Sur une équation fonctionnelle. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske Ser. II. **12** (1957), 201–205. (Serbo-Croatian summary)

Let S and S' be two euclidean spaces, E, $E_j\subseteq S$ ($j=1, \cdots, q$), $\bar{E}=E\cup(\bigcup E_j)$, $F\subseteq S'$, F compact; let g_j ($j=1, \cdots, q$) be maps of E onto $E_j=g_j(E)$, f a map of $\bar{E}\times F^q$ into F, such that $\|f(z, t)-f(z, t')\|\leqq M\|t-t'\|$ ($t, t'\in F^q$; $z\in\bar{E}$), where $\|t-t'\|=\sum_{j=1}^{q}\|t_j-t_j'\|$ ($0<Mq<1$).

The author gives a constructive proof for the solutions of the functional equation

$$Q(z)=f\{z, Q[g_1(z)], \cdots, Q[g_q(z)]\}\quad (z\in E),$$

demonstrates that there always exists at least one solution, shows that a sufficient condition for uniqueness is $E\supseteq\bigcup E_j$, and presents a necessary and sufficient condition for uniqueness in case $E\not\supseteq\bigcup E_j$.

E. F. Beckenbach (Los Angeles, Calif.)

SEQUENCES, SERIES, SUMMABILITY

See also 3829, 4191.

4111:

Rodero Carrasco, Julián. Special series. Gac. Mat., Madrid, **9** (1957), 170–180. (Spanish)

Expository.

4112:

Fog, David. A remark on two series. Nordisk Mat. Tidskr. **6** (1958), 83, 96. (Danish. English summary)

The classical series for $\pi/4$ and for ln 2 are particular

cases, with $p=1$ and $p=2$ respectively, of the expansion

$$\int_0^{\frac{\pi}{4}} \operatorname{tg}^{p-1} x\,dx = \frac{1}{p} - \frac{1}{p+2} + \frac{1}{p+4} - \frac{1}{p+6} + \cdots, \ p \geq 1.$$

Author's summary

4113:

Rényi, Alfréd. On algorithms for the generation of real numbers. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7** (1957), 265–293. (Hungarian)

4114:

Volinez, Gdalya. On the maximal sum of complex numbers when subjected to choice of signs. Riveon Lematematika **11** (1957), 23–25. (Hebrew. English summary)

Theorem: Let A be the class of all sequences $\{a_\nu\}$ where a_ν are complex numbers and $\sum |a_\nu|=1$. Let E be the class of all sequences $\{e_\nu\}$, $e_\nu=\pm 1$, $\nu=1, 2, 3, \cdots$. Then: $\operatorname{Inf}_A \operatorname{Sup}_E |\sum e_\nu a_\nu| = 2/\pi$. *Author's summary*

4115:

Topuriya, S. B. On a generalization of a theorem of Knopp. Soobšč. Akad. Nauk Gruzin. SSR. **19** (1957), no. 4, 385–392. (Russian)

Let the double series $\sum_{i,k} a_{ik}$ be given, and let

$$S_{mn} = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik}, \quad \sigma_{mn} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{k=1}^{n} S_{ik}.$$

The author establishes the following theorem. If

$$\sup_{\substack{1\leq m<\infty\\ 1\leq r\leq m}} \sum_{i=r}^{m} a_{in} < \frac{c}{n}, \quad \sup_{\substack{1\leq n<\infty\\ 1\leq s\leq n}} \sum_{k=s}^{n} a_{mk} < \frac{c}{m},$$

and if $\lim_{m,n\to\infty} \sigma_{mn}=S$, then $\lim_{m,n\to\infty} S_{mn}=S$. A number of corollaries of this theorem are stated.

H. P. Thielman (Ames, Iowa)

4116:

Labutin, D. N. Comparison of mean values. Kabardin. Gos. Ped. Inst. Uč. Zap. **12** (1957), 72–74. (Russian)

Let c_1, g_1, G_1 denote, respectively, the arithmetic, geometric and harmonic mean of two given positive numbers. Let c_n, g_n, G_n be defined (inductively) as the three means of the three numbers c_{n-1}, g_{n-1}, G_{n-1}. The author proves that $c_n-g_n>g_n-G_n>0$. A generalization to more than two numbers is given.

4117:

Ramanujan, M. S. The 'translativity' problem for quasi-Hausdorff methods of summability. Proc. Nat. Inst. Sci. India. Part A. **24** (1958), 4–14.

A transformation of the series $\sum u_n$ into the series $\sum v_n$, defined by $v_n=\sum_{k=0}^\infty h_{n,k}u_k$, is said to be conservative and the matrix $H=(h_{n,k})$ to be a δ-matrix if $\sum v_n$ converges to a limit whenever $\sum u_n$ does so. If, in addition, the two limits are the same, the transformation is said to be regular and the matrix an α-matrix. A δ-matrix or an α-matrix H for which $\lim_{k\to\infty} h_{n,k}=0$ is called a δ_0 matrix or α_0-matrix, respectively. Any matrix transforming $\sum u_n$ into $\sum v_n$ is said to be translative if, whenever $u_0+u_1+u_2+\cdots$ is summable to l by the matrix, then so is $0+u_0+u_1+\cdots$, and conversely. The matrix is translative to the left, or translative to the right, according as the first half alone of the condition of translativity or the second (converse) half alone of the condition is satisfied.

The problem of translativity for the Hausdorff matrices has been discussed by Kuttner [Proc. London Math. Soc. (3) **6** (1956), 117–138; MR **17**, 359]. In theorem I of the present paper, the author proves the following result for a quasi-Hausdorff matrix (H^*, μ_n) which is also a δ_0-matrix: Let (H^*, μ_n) and $(H^*, \mu_{n+1}/\mu_n)$ both be δ_0-matrices. Then, for the class **B** of series with bounded partial sums, (H^*, μ_n) is translative to the left. If, in addition, the limit constant associated with $\{\mu_{n+1}/\mu_n\}$ is not $\frac{1}{2}$, then (H^*, μ_n) is translative for the class **B**.

The result is similar to that for Hausdorff methods given by Kuttner in theorem 2 of his paper referred to above; the proof is based on eight lemmas. Translativity theorems of Vermes [Amer. J. Math. **71** (1949), 541–562; MR **10**, 699; theorems 3.III and 3.IV] for the method $A(p)$ of Taylor series continuation, where the boundedness of the partial sums of the series is not required, are deduced by the author from the proof of his theorem I.

Theorem II is an unpublished result by Kuttner, and shows how a deduction from theorem I for bounded sequences can be improved: If (H^*, μ_{n+1}) is any conservative sequence-to-sequence quasi-Hausdorff transformation, then (H^*, μ_{n+1}) is translative for bounded sequences. *R. G. Cooke* (London)

4118:

Rubin, L. A. A comparison of non-linear methods of summation of series with the methods of Cesàro. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. **5**, 102–111 (1957). (Russian)

Definition: $s_n\to s(\phi)$, if $\phi_m=\sum_{n=0}^\infty c(n, m)|s_n-s|^{k(n,m)}$ converges for $m=0, 1, 2, \cdots$ and $\phi_m\to 0$ as $m\to\infty$. Here the c's are non-negative real numbers, $C_m=\sum_n c(n, m)<\infty$ and the k's are real numbers in an interval $1\leq k\leq M$, where M is independent of n and m. By using simple functional analysis, the author shows that the ϕ-method is regular if and only if: 1) the C_m are uniformly bounded; 2) $c(n,m)\to\gamma(n)$ as $m\to\infty$; 3) if $\gamma(n)\neq 0$, then $k(n, m)\to k(n)$ as $m\to\infty$.

A method of summability similar to the ϕ-method is compared to the $(C, 1)$-method.

W. H. J. Fuchs (Ithaca, N.Y.)

4119:

✶Zeller, Karl. Theorie der Limitierungsverfahren. Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 15. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. viii+242 pp. DM 36.80.

Die ersten 166 Seiten des Buches sind in die folgenden Kapitel eingeteilt: Grundbegriffe der Limitierung, Hilfsmittel aus der Funktionalanalysis, Struktur von Wirkfeldern, Direkte Sätze, Umkehrsätze, Verfahren vom Cesàro-Abel Typ, Verfahren funktionentheoretischen Typs, Weitere Verfahren und Klassen. Es folgen 64 Seiten Literaturverzeichnis (vergleichbar den Registern der Referatenorgane nach Jahren geordnet) und Verzeichnisse der Bezeichnungen und Sätze.

Das Buch will einen Überblick über die Theorie der Limitierungsverfahren seit ihren Anfängen (etwa 1880) geben. Dies bedeutet, wie der Verf. in der Einleitung bemerkt, dass nur typische und leicht verständliche Sätze formuliert werden können; von den Beweisen werden höchstens die Grundzüge angegeben. Ferner beschränkt sich der Verf. auf Matrixverfahren und gewöhnliche Limitierung; Fragen über absolute und starke Limitierung, sowie Integraltransformationen werden nur kurz gestreift.

Die ersten 4 Kapitel des Buches behandeln die allgemeine Theorie der Matrixverfahren. Zunächst werden die verwendeten Hilfsmittel aus der Funktionalanalysis aufgeführt. Anschliessend wird in die Wirkfelder eine

Topologie eingeführt, wodurch funktionalanalytische Überlegungen zur Anwendung kommen können und zur Definition der Perfektheit und Abschnittskonvergenz führen. Es folgt die Erörterung zahlreicher Einzelfragen, wie Einfolgenverfahren, vorgeschriebenes Wirkfeld, Inäquivalenzsätze, Konvergenzfaktoren, Vergleichssätze, Verträglichkeit. Es wäre wünschenswert gewesen, die (grösstenteils in Buchform vorhandene) Beschreibung der funktionalanalytischen Hilfsmittel kürzer zu fassen zugunsten einer eingehenderen und ausgeglicheneren Darstellung der neueren Anwendungen in Kapitel III und IV, die den wesentlichen Beitrag dieses Buches ausmachen. Der Verf. versucht hier (und später) statt Ergebnissen allgemeine Prinzipien in den Vordergrund zu stellen, von denen aber einige in ihrer Formulierung trivial wirken (z.B. 35 VI, 36 II, 38 I,II, 45 I,II mit $\delta_m=1/m^2$).

In den verbleibenden 4 Kapiteln werden in grösster Kürze Umkehrsätze, spezielle Verfahren, analytische Fortsetzung und Anwendungen behandelt. Darüber liegen zumeist schon ausführlichere Darstellungen in Buchform vor (etwa Hardy-Riesz, Knopp, Hardy). Insbesondere wird die Darstellung der Taubersätze, die meist als Hauptergebnisse der Theorie der Limitierungsverfahren angesehen werden, der vorliegenden Literatur von über 300 Arbeiten kaum gerecht. Oft wird nur eine Literaturaufzählung gegeben, zum Teil (wie auf S. 116, 123, 146) ohne jedes Eingehen auf den Inhalt der zitierten Arbeiten. Man vermisst andererseits wichtige Dinge, wie z.B. die allgemeine Formulierung der analytischen Fortsetzung durch Lorentz [Bernstein polynomials, Univ. of Toronto Press, 1953; MR **15**, 217; S. 117–121], Summierbarkeit von Dirichletreihen, funktionentheoretische Taubersätze bei Dirichletreihen vom Typ Landau-Schnee, tiefere Multiplikationssätze [vgl. etwa Hardy-Riesz, oder Chandrasekharan-Minakshisundaram, Typical means, Oxford Univ. Press, 1952; MR **14**, 1077], Taubersätze aus der Theorie der Wahrscheinlichkeit und Ergodensätze [vgl. etwa Feller, An introduction to probability theory and its applications, Wiley, New York, 1950; MR **12**, 424], die Theorie der regulär wachsenden Funktionen von Karamata, einseitige Konvexitätssätze, Überkonvergenz, die Theorie der Mercersätze und Lückensätze von Pitt-Wiener, Umkehrsätze für die Stieltjestransformation im Zusammenhang mit asymptotischer Nullstellenverteilung ganzer Funktionen.

{Der Ref. bemerkt u.a. noch die folgenden Einzelheiten. Bei 14,I sollte die Arbeit in Bull. Amer. Math. Soc. **44** (1938), 91–93 von Bohnenblust-Sobczyk erwähnt werden. Bei 15 VI sind die Begriffe bilinear und linear auf dem Produktraum verwechselt. Bei den genauen Quellenangaben auf S. 41–44 hat der Verf. verschiedentlich (so bei (1), (2), I, II, III und auf S. 43, 44 oben) Arbeiten von Jurkat und dem Ref. [Math. Z. **55** (1951), 23–54, 92–108; **56** (1952), 152–178; MR **13**, 933, 934; **14**, 158] übersehen. Insbesondere sind auch die Aussagen auf S. 44, Zeilen 16, 26 und 27 v.o. umzukehren. Auf S. 62 ist in III die wichtige Tatsache nicht vermerkt, dass (2) auch ohne die Voraussetzung der Abschnittskonvergenz gilt. Durch II auf S. 121 werden praktisch nur äquivalente Verfahren erfasst (und z.B. nicht $p_m=\log q_m$). Auf S. 128–129 hätte sich durch die Verwendung von A_1^* — mit Singularitäten auf (0, 1) — statt A_1 ein besserer Verträglichkeitssatz ergeben (Verfahren wie $p_0=1$, $p_1=-2$, $p_2=\cdots=0$ werden hier nicht erfasst).} *A. Peyerimhoff* (Giessen)

4120:

Parameswaran, M. R. On some Mercerian theorems in summability. Proc. Amer. Math. Soc. **8** (1957), 968–974.

The author applies the following principle to Mercerian theorems: An element $I+qA$ in a Banach algebra has an inverse if $|q|\cdot\|A\|<1$. One of his results (theorem 4) states: If the lower-semimatrix P transforms every $x\in|c|$ into $|c|$ and satisfies

$$p_{nn}-\sum_{k=n+1}^{\infty}\Big|\sum_{i=n}^{k}(p_{ki}-p_{k-1,i})\Big|>\lambda>0$$

for all $n=0, 1, \cdots$, then $Px\in|c|$ implies $x\in|c|$. Here $|c|$ is the set of all absolutely convergent sequences $x=\{x_n\}$, i.e., sequences satisfying $\sum|x_n-x_{n-1}|<\infty$. [Cf. Bosanquet, J. London Math. Soc. 13 (1938), 177–180; and Love, ibid. **27** (1952), 413–428; MR **14**, 159.]

Others of the author's theorems deal with common summability; the matrices in question transform either c_0 into c_0, c into c or m into m. [Cf. Širokov, Uspehi Mat. Nauk (N.S.) **10** (1955), no. 4(66), 167–170; MR **17**, 840. For more literature, see the reviewer's book, #4119 above, p. 78.] *K. Zeller* (Tübingen)

4121:

Kangro, G. F. Über Matrixtransformationen von Folgen in Banachschen Räumen. Izv. Akad. Nauk Èston. SSR. Ser. Tehn. Fiz.-Mat. Nauk **1956**, 108–128. (Russian. Estonian and German summaries).

Let X be a real Banach space, and c_X, m_X, and l_X the spaces of sequences $\{x_\lambda\}_{\lambda=0}^{\infty}$ $(x_\lambda\in X)$ such that $\lim x_\lambda$ exists in X strongly, $\|x_\lambda\|$ is bounded, and $\sum_\lambda\|x_\lambda\|<\infty$, respectively. Let Y be a real Banach space and $A_{n\lambda}$ $(n, \lambda=0, 1, 2, \cdots)$ bounded linear mappings of X into Y. Let (*) $y_n=\sum_\lambda A_{n\lambda}x_\lambda$ for a sequence $\{x_\lambda\}$ of elements of X $(n=0, 1, 2, \cdots)$. If the series converges strongly in Y, then (*) defines a sequence of elements of Y. The author finds necessary and sufficient conditions for the mapping (*) to carry m_X into c_Y, l_X into c_Y, and l_X into l_Y. For example: if $\lim_{n\to\infty}A_{n\lambda}=A_\lambda$ $(\lambda=0, 1, 2, \cdots)$ exists in norm, then (*) carries m_X into c_Y if and only if (1) $\sum_\lambda A_{n\lambda}x_\lambda$ converges for all $\{x_\lambda\}\in m_X$, and (2)

$$\lim_{n\to\infty}\sup_{\|x_\lambda\|\le1}\Big\|\sum_{\lambda=0}^{p}(A_{n\lambda}-A_\lambda)x_\lambda\Big\|=0$$

uniformly in $p=0, 1, 2, \cdots$. If (1) and (2) hold, then we have $\lim_{n\to\infty}y_n=\sum_\lambda A_\lambda x_\lambda$, and the series converges uniformly for $\|x_\lambda\|\le1$.

[For mappings (*) carrying c_X into c_Y, see Melvin-Melvin, Proc. London Math. Soc. (2) **53** (1951), 83–108; MR **13**, 45; and Zeller, Math. Z. **56** (1952), 18–20; MR **14**, 158; and for the classical case in which $X=Y=$the real numbers, see the recent book of Zeller, #4119 above]. *Edwin Hewitt* (Seattle, Wash.)

4122:

Kangro, G. F. On linear and bilinear transformation of sequences in Banach space. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 199–201. (Russian)

Let X and Z be Banach spaces and let A_{nk} $(n, k=0, 1, 2, \cdots)$ be bounded operators on X to Z. The relation

$$z_n=\sum_{k=0}^{\infty}A_{nk}x_k$$

defines a transformation of certain sequences of elements of X into sequences in Z. If this transformation maps all elements of a given class $\mathfrak{X}$ of sequences in X into a class $\mathfrak{Z}$ of sequences in Z, then it is said to possess property $\mathfrak{X}\to\mathfrak{Z}$. Similarly, let Y be a third Banach space and let

A_{nkl} ($n, k, l=0, 1, 2, \cdots$) be bounded operators on the Cartesian product $X\times Y$ to Z. Also let $\mathfrak{Y}$ be a class of sequences in Y. The relation

$$z_n=\sum_{k,l=0}^{\infty} A_{nkl}x_ky_l$$

defines a transformation of certain pairs of sequences $\{x_k\}\subset X$, $\{y_l\}\subset Y$ into sequences in Z. This transformation is said to possess property $\mathfrak{X}, \mathfrak{Y}\to\mathfrak{Z}$ if the pair $\{x_k\}\in\mathfrak{X}$ and $\{y_l\}\in\mathfrak{Y}$ is transformed into $\mathfrak{Z}$.

For any Banach space X, denote by c_X all convergent sequences, m_X all bounded sequences and l_X all absolutely convergent series of elements of X. Necessary and sufficient conditions for the property $c_X\to c_Z$ were obtained by Melvin-Melvin [Proc. London Math. Soc. (2) **53** (1951), 83–108; MR **13**, 45], and for the properties $m_X\to c_Z$, $l_Y\to c_Z$, $l_X\to l_Z$ by the author [Izv. Akad. Nauk Eston. SSR. Ser. Tehn. Fiz.-Mat. Nauk **1956**, 108–128; #4121 above]. In the present paper the author states necessary and sufficient conditions for the property c_X, $l_Y\to c_Z$. He also indicates some applications of the earlier results as well as this result to summability of series of elements in a Banach space. *C. E. Rickart* (New Haven, Conn.)

4123:

Włodarski, L. Sur les méthodes continues de limitation du type de Borel. Ann. Polon. Math. **4** (1958), 137–164.

Verfasser untersucht die Verfahren B_k, die einer Folge $\{\xi_n\}$ den Grenzwert $\lim_{t\to\infty} 2^k e^{-t}\sum_n \xi_n\cdot t^{n2^k}/\Gamma(n2^k+1)$ zuordnen (falls dieser vorhanden ist). Für k sind die Werte $0, \pm1, \pm2, \cdots$ zugelassen; bei $k=0$ ergibt sich das gewöhnliche Borelverfahren. Verfasser beweist unter anderem folgenden Satz. Th. 4: Existiert B_p-lim $\xi_n=\xi$ sowie die B_q-Transformation von $\{\xi_n\}$ (wo $q<p$), so existiert auch B_q-lim $\xi_n=\xi$. Ein entsprechender Satz gilt, wenn man B_q durch das Abelverfahren ersetzt (Th. 5). Die B_k sind rechtstranslativ (Th. 2), die Wirkfelder der B_k alle verschieden (Th. 9). Ferner bestimmt Verfasser B_k-Limitierbarkeitsbereiche der geometrischen Folge (Th. 6 und 7). {Eine ausführlichere Auseinandersetzung mit schon bekannten Resultaten über verallgemeinerte Borel-Verfahren wäre wünschenswert gewesen; siehe Literatur bei Zeller [#4119 obenstehend], Seite 140.} *K. Zeller* (Tübingen)

4124:

Hsiang, Fu Cheng. On the convergence criterion of an oscillating series. Bull. Calcutta Math. Soc. **47** (1955), 203–207.

Let $\varphi(t)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n\cos nt$. The following convergence test is proved: Let $0\leqq\lambda<1$. If (i) $\varphi(t)=o\{\log 1/t\}^{-\lambda}$ as $t\to0$, and (ii) $a_n>-K(\log n)^{\lambda}/n$ ($K>0$), then $\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n$ converges to the sum 0. It is first shown that (i) implies $\sigma_n=o(\log n)^{-\lambda}$, where σ_n denotes the arithmetic means {the proof given is unnecessarily complicated}. Condition (ii) is then used in the application of a Tauberian theorem which is interesting in itself. A function $f(x)>0$ for $x\geqq0$ is said to be of class K if $f(cx)/f(x)\leqq M$, where $\frac{1}{2}\leqq c<2$. The theorem is: Let $\varphi(x)$ and $\psi(x)$ be of class K, and $\varphi(n)=O(n^2\psi(n))$. If $u_n=o(\varphi(n))$ and $\Delta^{r+1}u_n>-k\psi_n$, then $\Delta^r(u_n)=o\{[\varphi(n)\cdot\psi(n)]^{\frac{1}{2}}\}$. In the application, $r=1$, $u_n=n\sigma_n$, $\varphi(x)=(x+2)(\log(x+2))^{-\lambda}$, and $\psi(x)=(x+2)^{-1}(\log(x+2))^{\lambda}$. *W. W. Rogosinski* (Zbl **66**, 304)

4125:

Lamperti, John. On the coefficients of reciprocal power series. Amer. Math. Monthly **65** (1958), 90–94.

Let C be the class of all power series $U(z)=u_0+u_1z+u_2z^2+\cdots$ (with $u_0=1$) with the property that the power series for $F(z)=1-(U(z))^{-1}$ has non-negative coefficients only. Some sufficient conditions for this property were given by T. Kaluza [Math. Z. **28** (1928), 161–170], e.g., the condition that $u_1>0$, $u_{n-1}u_{n+1}\geqq u_n^2$ ($n=1, 2, \cdots$). The present author gives an elegant necessary and sufficient condition: we have $U\in C$ if and only if there exists a matrix $P=(p_{ij})$ ($i, j=1, 2, \cdots$), with $p_{ij}\geqq0$ for all i, j, and with only a finite number of non-zero terms in each row, such that $u_n=p_{11}^{(n)}$ ($p_{ij}^{(n)}$ denote the elements of the nth power of P). As applications, some theorems of the following type are given: if $\sum u_nz^n$ and $\sum v_nz^n$ both belong to C, then the Hadamard product $\sum u_nv_nz^n$ lies in C. It is also derived that $U\in C$ implies $u_{n+m}\geqq u_nu_m$ for all n and m. *N. G. de Bruijn* (Amsterdam)

APPROXIMATIONS AND EXPANSIONS

See also 4003, 4026.

4126:

Berman, D. L. Divergence of the Hermite-Féjer interpolation process. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 2(80), 143–148. (Russian)

Let $x_k=-1+2k/n$, $k=0, 1, \cdots, n$, be equidistant nodes in $[-1, +1]$, $H_n(x)$ be the interpolation polynomial of degree $2n+1$ which coincides with x in the points x_k, but has derivative equal to zero. Then $H_n(x)$ diverges unboundedly for $n\to\infty$ at each point of $[-1, +1]$ except the point $x=0$. *G. G. Lorentz* (Syracuse, N.Y.)

4127a:

de Albuquerque, Luís G. M. Fundamental problems of the theory of functional approximation. II. Gaz. Mat., Lisboa, **18** (1957), 24–28. (Portuguese)

4127b:

de Albuquerque, Luís G. M. Fundamental problems of the theory of functional approximation. III. Gaz. Mat., Lisboa, **19** (1958), 18–23. (Portuguese)

Expository. For part 1 see same Gaz. **18** (1957), no. 66–67, 18–28 [MR **19**, 667].

4128:

Ganguli, P. L. A note on Ostrowski's generalization of a theorem of Osgood. Bull. Calcutta Math. Soc. **49** (1957), 75–78.

Osgood [Amer. J. Math. **19** (1897), 155–190; p. 161] considered the convergence of a series $\sum a_n(x)$ where $a_n(x)$ is continuous and where the sum function is also continuous. It is well known that the convergence is not necessarily uniform. Osgood was making a study of the approximation curves. Ostrowski proved the following theorem, which is essentially an extension to continuous functions of a result by Osgood for sequences.

Theorem: Let $f(t, x)$ be a continuous function of the point (t, x) for $a<x<b$ and $t\geqq T$, and suppose that we have $\lim_{t\to\infty} f(t, x)=f(x)$, where $f(x)$ is also continuous in (a, b). Then, for every $\varepsilon>0$, there exists a subinterval J of (a, b) and a number T_0 such that, for $x\in J$, $t\geqq T_0$, we have $|f(t, x)-f(x)|<\varepsilon$. [Proc. Amer. Math. Soc. **1** (1950), 648–649; MR **13**, 121].

The present note reduces Ostrowski's assumptions by assuming $f(t, x)$ continuous in x only. *T. Fort* (Columbia, S.C.)

4129:

Verblunsky, S. On an expansion in exponential series. Quart. J. Math. Oxford Ser. (2) 7 (1956), 231–240.

Let $k(u)$ be a function of bounded variation, possibly complex, defined in the interval (0, 1). Let λ_ν, $\nu=1, 2, \cdots$, be the zeros of the integral function $A(z)=\int_0^1 k(u)e^{zu}du$. Let $k(0+)k(1-)\neq 0$, $A'(\lambda_\nu)\neq 0$, $A(0)\neq 0$, $0<|\lambda_1|\leq|\lambda_2|\leq\cdots$. Let

$$\varphi_\nu(u)=\frac{k(u)}{\lambda_\nu}+e^{-\lambda_\nu u}\int_u^1 k(v)e^{\lambda_\nu v}dv \quad (\nu=1, 2, \cdots),$$

and, for a function $f(u)\in\mathscr{L}(0, 1)$, put $\alpha_0=A(0)^{-1}$. $\int_0^1 f(u)k(u)du$, $\alpha_\nu=A'(\lambda_\nu)^{-1}\int_0^1 f(u)\varphi_\nu(u)du$ $(\nu=1, 2, \cdots)$. There exists a $\delta>0$ and an unbounded increasing sequence of positive numbers r_p such that the circle $|z|=r_p$ has no points in common with any of the disks of radius δ around the zeros of $zA(z)$. There is a wide choice for the r_p, e.g., they can be chosen so that $r_{p+1}-r_p\leq 1$. Let ν_p denote the greatest integer ν such that $|\lambda_\nu|<r_p$.

The following two theorems are proved. Special cases have previously been considered by J. Delsarte [J. de Math. **14** (1938), 403–53]. Theorem 1. Let $f(u)\in\mathscr{L}(0, 1)$. Then, as $p\to\infty$,

$$\alpha_0+\sum_1^{\nu_p}\alpha_\nu e^{\lambda_\nu t}-\frac{1}{\pi}\int_0^1 f(u)\frac{\sin r_p(t-u)}{t-u}du$$

converges to zero uniformly in any closed interval (α, β), interior to the open interval (0, 1). Theorem 2. Let $f(u)$ be integrable in every finite interval, and satisfy the equation

$$\int_0^1 k(u)f(t+u)du=0$$

for all t. Let (a, b) be an assigned finite interval. Then, as $p\to\infty$,

$$\sum_1^{\nu_p}\alpha_\nu e^{\lambda_\nu t}-\frac{1}{\pi}\int_a^b f(u)\frac{\sin r_p(t-u)}{t-u}du$$

converges to zero uniformly in any closed interval interior to the open interval (a, b). *E. Følner* (Copenhagen)

4130:

Shao, Pin-tsung. An estimation about the approximation by Bernstein polynomials of dispersiveness for a continuous function. Advancement in Math. 4 (1958), 282–287. (Chinese. English summary)

If $\{r_n\}$ is a strictly increasing sequence of positive integers with $r_n-r_{n-1}=O(r_n^{\alpha})$, where $0\leq\alpha<\frac{1}{2}$, it is proved that

$$\beta_{f,r_n}(x)=f(x)+O(\omega(r_n^{-\frac{1}{2}+\varepsilon}))+O(r_n^{-\frac{1}{2}+\alpha+\varepsilon});$$

with $\varepsilon>0$ and

$$\omega(\delta)=\sup_{|x'-x''|\leq\delta}|f(x')-f(x'')|,$$

where

$$\beta_{f,r_n}(x)=\sum_{k=0}^{n}f\left(\frac{r_k}{r_n}\right)\binom{r_k}{r_n}(r_{k+1}-r_k)x^{r_k}(1-x)^{r_n-r_k}$$

is the so-called Bernstein polynomial of dispersiveness [see, e.g., G. G. Lorentz, Bernstein polynomials, Univ. of Toronto Press, Toronto, Ont., 1953; MR **15**, 217].

From the author's summary

4131:

Mangeron, D. Sur une nouvelle classe de polynomes intéressant les équations aux "dérivées totales". Bul. Inst. Politehn. Iaşi (N.S.) 2 (1956), 21–27. (Romanian. Russian and French summaries)

4132:

Hsu, L. C. Concerning best approximations to certain classes of functions by Arnold's type of singular integrals. Advancement in Math. 2 (1956), 695–702. (Chinese)

Let K be a real continuous function on $a\leq x\leq b$, $a\leq t\leq b$, having the following two properties: (1) $|K(x,t)|<K(x,x)$ for $t\neq x$; (2) for each x with $a<x<b$, there exists $h=h(x)>0$ such that $\lim_{t\to x}|K(x, t)-K(x, x)|\cdot|t-x|^{-h}>0$. Let $J_n(x)=\int_a^b[K(x,t)]^n dt$; $P_n(f;x)=[J_n(x)]^{-1}\int_a^b[K(x,t)]^n f(t)dt$. For functions f of class C or C^1 or functions f satisfying a Lipschitz condition, the author studies the asymptotic behavior of $P_n(f; x)-f(x)$, as $n\to\infty$. A typical result states that $P_n(f; x)-f(x)=o(n^{-1/h})$, provided $f\in C^1$ and $f(x)\neq 0$. This paper is related to one by G. A. Arnol'd [Mat. Sb. N.S. **29(71)** (1951), 427–432; MR **13**, 341], but the hypothesis concerning K in the present paper is much less restrictive. Also certain results of I. P. Natanson [Dokl. Akad. Nauk SSSR **45** (1944), 274–277; **54** (1946), 11–13; MR **6**, 267; **8**, 577] concerning the singular integral of de La Vallée-Poussin are generalized.

Ky Fan (Notre Dame, Ind.)

4133:

Bugrov, Ya. S. Approximation by trigonometric polynomials of classes of functions defined by a polyharmonic operator. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 2(80), 149–156. (Russian)

Let Λ_r^p denote the class of 2π-periodic functions $f(x, y)$ which have derivatives of all orders not exceeding $2r$, with the property that $\varphi=\Delta^p f$ has L^p norm $\|\varphi\|_p\leq 1$ in $0\leq x, y\leq 2\pi$. The author considers $A_{mn}=\sup\|f-S_{mn}(f)\|_p$, where S_{mn} is the partial sum of the Fourier series of f, and the supremum is taken for all $f\in\Lambda_r^p$. He proves that, for $p=1$ and for $p=\infty$, $A_{mn}=16\pi^{-4}(m^2+n^2)^{-r}\log m\log n+O(m^{-2r}\log m+n^{-2r}\log n)$. He also studies the degree of approximation by trigonometric polynomials of functions of the class Λ_r^p. *G. G. Lorentz* (Syracuse, N.Y.)

4134:

Békéssy, András. Eine Verallgemeinerung der Laplaceschen Methode. Magyar Tud. Akad. Mat. Kutató Int. Közl. 2 (1957), 105–125. (Hungarian and Russian summaries)

Suppose that the following conditions are satisfied: (i) $x\geq 0$, $0<a\leq\infty$; (ii) $\exp[-xf(t)]$ is integrable over $(0, a)$ for sufficiently large x; (iii) for each $\delta>0$, $\inf f(t)$ for $\delta\leq t\leq a$ is positive; (iv) $f(t)\to 0$ as $t\to+0$; (v) g satisfies the same conditions as f; and (vi) $f(t)\sim g(t)$ as $t\to 0$. For special g, for instance, for $g(t)=ct^{\alpha}$, $\alpha>0$, it is known that under these circumstances (*) $\int_0^a\exp[-xf(t)]dt\sim\int_0^a\exp[-xg(t)]dt$ as $x\to\infty$ holds. The author shows that (i) to (vi) do not, in general, ensure the validity of (*) and gives two criteria, either of which, together with (i) to (vi) implies (*). His criteria demand the existence of a c, $0<c<1$, so that either

$$\text{(I)}\qquad \limsup\left\{\int_0^a\exp[-cxg(t)]dt\Big/\int_0^a\exp[-xg(t)]dt\right\}$$

is finite as $x\to\infty$, or

$$\text{(II)}\qquad \limsup[g(ct)/g(t)]<1,$$

as $t\to+0$. {The condition $c<1$ is inadvertently omitted in the text in criterion (I), but it appears in the Hungarian and Russian summaries.} These criteria are applied to prove three theorems on asymptotic representations of integrals. *A. Erdélyi* (Pasadena, Calif.)

4135:
Wall, H. S. Some convergence problems for continued fractions. Amer. Math. Monthly **64** (1957), no. 8, part II, 95–103.

Conditions for the convergence of a continued fraction are obtained when certain subsequences of the sequence of approximants converge. A number of other convergence theorems are also given. *E. Frank* (Chicago, Ill.)

FOURIER ANALYSIS

See also 4124, 4133.

4136:
✱Franklin, Philip. An introduction to Fourier methods and the Laplace transformation. Dover Publications, Inc., New York, 1958. x+289 pp. $1.75.

Unabridged and corrected republication, formerly entitled "Fourier methods", and published by McGraw-Hill, New York, 1949 [MR **11**, 429].

4137:
✱Jeffery, R. L. Trigonometric series. Canadian Mathematical Congress Lecture Series No. 2. University of Toronto Press, Toronto, 1956. 39 pp. $2.50.

Dieser vor einem nicht rein mathematischen Publikum gehaltene Vortrag entwickelt in dem ersten Teil die Problemgeschichte der Fourierschen Reihen von den Anfängen bis zur Gegenwart. Sodann werden in dem zweiten Teil die Beweise für die im geschichtlichen Überblick vorgetragenen Ergebnisse nachgeholt. Es wird sogar auf die Integralbegriffe von Denjoy und Perron und die Arbeiten von Marcinkiewicz und Zygmund, Burkill und James eingegangen. *V. Garten* (Zbl **72**, 285)

4138:
Chao, Chi-Chang. The closed-form summation of some common Fourier series. Quart. J. Mech. Appl. Math. **9** (1956), 508–512.

The author has given a method of summation in closed form of Fourier series whose coefficients are ratios of polynomials of certain type. This method is valid only for sine and cosine series. The closed form summation of Fourier series of this type is obtained by solving an ordinary differential equation. The form of this differential equation is determined by the denominator of the coefficients of the given Fourier series and the boundary conditions satisfied by the differential equations are determined by the numerator of the coefficients. *U. N. Singh* (Zbl **72**, 62)

4139:
Izumi, Shin-ichi; and Satô, Masako. Fourier series. I. Parseval relation. Proc. Japan Acad. 32 (1956), 446–450.

Verff. geben hinreichende Bedingungen für das Bestehen der Parsevalschen Gleichung

$$(1)\qquad \frac{1}{\pi}\int_0^{2\pi} f(x)g(x)dx=\tfrac{1}{2}a_0a_0'+\sum_{n=1}^{\infty}(a_na_n'+b_nb_n')$$

in der folgenden Form: 1. $g(t)$ sei L-integrabel und $f(t)$ sei eine beschränkt meßbare Funktion derart, daß

$$N\int_x^{x+\pi/N}\left|\sum_{k=1}^{[N/2]}\frac{f(t\pm(2k-1)\pi/N)-f(t\pm 2k\pi/N)}{k}\right|dt$$

in x gleichmäßig beschränkt ist und fast überall (nicht notwendig gleichmäßig in x) gegen Null strebt; dann gilt (1). 2. $g(t)$ sei beschränkt meßbar und $f(t)$ sei eine integrable Funktion derart, daß

$$\int_0^{2\pi}\left|\sum_{k=1}^{[N/2]}\frac{f(t\pm 2k\pi/N)-f(t\pm(2k-1)\pi/N)}{k}\right|dt$$

für $N\to\infty$ gegen Null strebt; dann gilt (1). Drei weitere Bedingungen werden ohne Beweis mitgeteilt.
V. Garten (Zbl **72**, 286)

4140:
Hsiang, Fu Cheng. On a test for the convergence of Fourier series. Proc. Amer. Math. Soc. **7** (1956), 1036–1039.

It was proved by F. T. Wang [Proc. London Math. Soc. (2) **47** (1942), 308–325; MR **4**, 37] that the Fourier series of an even function $\phi(t)$ converges at $t=0$ if $\int_0^t\phi(u)du=o(t/\log t^{-1})$, $t\to+0$, and if the Fourier coefficients a_n satisfy the inequalities $a_n>-K(\log n)/n$, where K is a positive constant. Also, if these inequalities are replaced by the condition $a_n=o\{(\log n)^2/n\}$, then the theorem is false.

In the article under review it is shown that the theorem of Wang is already false if the above inequalities are replaced by the condition $a_n=o\{(\log n)^{1+\eta}/n\}$, where η is an arbitrary positive number.
A. A. Šneider (RŽ Mat **1957** #6960)

4141:
Menš'ov, D. E. Limits of indeterminacy in measure of partial sums of trigonometric series. Advancement in Math. 3 (1957), 547–561. (Chinese)

Theorem. Let F and G be defined and measurable on $[-\pi,\pi]$ and $F\leqq G$ there. Then there exists a trigonometric series $\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ with $a_n\to 0$ and $b_n\to 0$ such that for any increasing sequence of integers $\{n_k\}$, F and G are, respectively, the upper and lower limits in measure [for definition see Menš'ov, Trudy Mat. Inst. Steklov 32 (1950); MR 12, 254] of the sequence of partial sums S_{n_k} where S_n is the partial sum of the above series to n terms. *K. L. Chung* (Syracuse, N.Y.)

4142:
Mohanty, R.; and Izumi, Shin-ichi. On the absolute logarithmic summability of Fourier series of order one. Tôhoku Math. J. (2) **8** (1956), 201–204.

Suppose that $0\leqq\lambda_0\leqq\lambda_1\leqq\cdots\leqq\lambda_n\leqq\cdots$, that $\lambda_n\to\infty$ and that $\sum_n A_n$ is a given series. By definition, this series is summable $|R,\lambda_n,1|$ if

$$\sum_n\left\{\frac{1}{\lambda_n}-\frac{1}{\lambda_{n+1}}\right\}\left|\sum_{m=1}^{n}\lambda_mA_m\right|<+\infty.$$

The present article considers the case $\lambda_n=\log(n+1)$. The authors have shown earlier, independently of each other [S. Izumi, same J. (2) **1** (1950), 136–143; MR **12**, 254; R. Mohanty, J. London Math. Soc. **25** (1950), 67–72; MR **11**, 592], that the property of the Fourier series of $f(t)$ of being summable $(R,\log n,1)$ at the point x is not a local property, but depends on the behavior of $f(t)$ in the whole interval $(-\pi,\pi)$.

The present article considers the conditions under which this property is local. Suppose that $\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}$ and that $\sum A_n\cos nt$ is the Fourier series of $\phi(t)$. We have: 1) if $\sum_n|A_n|n^{-1}\log\log n<+\infty$, then the summability $|R,\log n,1|$ of the Fourier series of $f(t)$ at the point x depends only on its behavior about $t=x$; 2) if $\int_0^{2\pi}|f(t)+f(t+\pi/n)|dt=O(\log\log n)^{-1-\delta}$, $\delta>0$, then the summability $|R,\log n,1|$ of the Fourier series of $f(t)$ is a local property. *N. K. Bari* (RŽ Mat **1957** #6963)

4143:

Bhatt, S. N. The absolute summability (A) of Laplace series. Bull. Calcutta Math. Soc. **49** (1957), 129–132.

The main result is as follows. Let $f(\theta, \psi)$ be integrable (L) over the unit sphere, with the coordinates θ, ψ chosen so that a given point P on the sphere is the pole. Define $\varphi_\alpha(\theta)=\Gamma(\alpha+1)\theta^{-\alpha}\phi_\alpha(\theta)$ ($\alpha\geqq 0$) where $\phi_\alpha(\theta)=[1/\Gamma(\alpha)]\int_0^\theta (\theta-u)^{\alpha-1}\varphi(u)du$ for $\alpha>0$, and $\phi_0(\theta)=\varphi(\theta)$, with $\varphi(\theta)=\int_0^{2\pi} f(\theta, \psi)d\psi-\int_0^{2\pi} f(0, \psi)d\psi$. If, for some $\alpha\geqq 0$ and some $\delta>0$, $\varphi_\alpha(\theta)$ is of bounded variation in $(0, \delta)$ and $\varphi_\alpha(\theta)\to 0$ for $\theta\to 0$, then the Laplace series

$$(4\pi)^{-1}\sum_{m=0}^{\infty}(2m+1)\int_0^{\pi}\int_0^{2\pi} f(\theta, \psi)P_m(\cos\theta)\sin\theta \, d\theta d\psi,$$

where P_m are polynomials of Legendre of the first class, is absolutely summable (A) to the sum $f(P)$.

A. G. Azpeitia (Amherst, Mass.)

4144:

Pokalo, A. K. A theorem on summation of series of functions of $W^{(r)}$ classes. Minsk. Gos. Ped. Inst. A. M. Gor'k. Uč. Zap. **7** (1957), 51–65. (Russian)

Study, along the lines of the investigation reviewed above [#4003], of means $\sum_{k=0}^{n}\mu_n(k)(a_k\cos kx+b_k\sin kx)$ of functions $f(x)\sim\sum(a_k\cos kx+b_k\sin kx)$, with $|f^{(r)}(x)|\leqq 1$.

W. H. J. Fuchs (Ithaca, N.Y.)

4145:

Ul'yanov, P. L. Divergence of Fourier series. Advancement in Math. **4** (1958), 198–248. (Chinese)

A translation of the Russian article in Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 75–132 [MR **19**, 854].

4146:

Kawata, Tatsuo. Mean convergence of a Fourier series and a Fourier transform. Kōdai Math. Sem. Rep. **7** (1955), 71–78.

If $f(x)\in L(-\pi, \pi)$ and $\in L_r(a, b)$, where $r>1$ and (a, b) is a subinterval of $(-\pi, \pi)$, in general, it cannot be asserted that the Fourier series of f converges in the mean $L_r(a, b)$, though this is true in $L_r(a+\varepsilon, b-\varepsilon)$ for a fixed small positive ε. With additional restrictions on the behaviour of f in the neighbourhoods of a and b, the desired result can be proved. The precise theorem proved by the author is as follows: Let $f\in L_p(-\pi, \pi)$, $p\geqq 1$, and have period 2π. Let for some $r\geqq p$, $f\in L_r(a, b)$ where (a, b) is subinterval of $(-\pi, \pi)$. Let for some $\alpha>1-1/r$, and constants s_1 and s_2, the relations $\int_0^t |f(b+x)-s_1|dx=O(t^\alpha)$ and $\int_0^t |f(a-x)-s_2|dx=O(t^\alpha)$ hold as $t\to+0$. Then the Fourier series of f converges in the mean $L_r(a, b)$ to $f(x)$. A similar result is proved for Fourier transforms.

V. Ganapathy Iyer (Zbl **67**, 43)

4147:

McMillin, Kenneth M. Abel summability of the double series successively derived from the double Fourier series. Tôhoku Math. J. (2) **8** (1956), 183–187.

The results of M. L. Misra [Duke Math. J. **14** (1947), 167–177; MR **8**, 577] are generalized. Namely, suppose that $f(x, y)$ is a periodic function, integrable over the square $[-\pi\leqq x\leqq\pi, -\pi\leqq y\leqq\pi]$ and

$$A(x, y, r, p)=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u, v)P(r, u-x)P(p, v-y)dudv,$$

where $0\leqq r<1$, $0\leqq p<1$ and $P(r, \phi)$ is the Poisson kernel; that is, $P(r, \phi)=\frac{1}{2}(1-r^2)[1+r^2-2r\cos\phi]^{-1}$. Sufficient conditions on $f(x, y)$ are derived in order that, at the point (x_0, y_0), the limit

$$\lim_{\substack{r\to 1\\ p\to 1}}\left.\frac{\partial^{j+k}A(x, y, r, p)}{\partial x^j\partial y^k}\right|_{(x_0, y_0)}$$

should exist.

P. L. Ul'yanov (RŽ Mat 1957 #8553)

4148:

Rudin, Walter. Transformations des coefficients de Fourier. C. R. Acad. Sci. Paris **243** (1956), 638–640.

Let $\sum c_n e^{in\theta}$ be the Fourier series of an absolutely integrable function. Then the author shows that for $\sum\varphi(c_n)e^{in\theta}$ to be the Fourier series of an absolutely integrable function, it is sufficient for $\varphi(z)$ to be representable by the absolutely convergent series $\varphi(z)=\varphi(x, y)=\sum a_{r,s}x^r y^s$ ($z=x+iy$, $a_{00}=0$) near the origin, and it is necessary for $\varphi(z)$ to satisfy the Lipschitz condition of order 1 near the origin. This theorem belongs to the same circle of ideas as Wiener-Levy's theorem [cf. I. Gelfand, Mat. Sb. N.S. **9** (1941), 49–50, 51–66; MR **3**, 36, 51].

G. Sunouchi (Zbl **71**, 60)

4149:

Kahane, J. P. Sur certaines classes de séries de Fourier absolument convergentes. J. Math. Pures Appl. (9) **35** (1956), 249–259.

Let A denote the class of functions $f(x)=\sum_{-\infty}^{\infty}a_n e^{inx}$ with $\|f\|=\sum|a_n|<\infty$, and $A\{\omega_n\}$ denote the class of functions $F(x)=\sum_{-\infty}^{\infty}A_n e^{inx}$ with $|||F|||=\sum|A_n|\omega_n<\infty$, where $\{\omega_n\}_{-\infty}^{\infty}$ is a given sequence such that $\omega_n\geqq 1$ for all n. A and $A\{\omega_n\}$ are Banach spaces with the norms $\|f\|$ and $|||F|||$ respectively. A^* denotes the class of functions f which are locally equal to functions of A. The author begins by proving the following extension of a theorem of Leibenson: If $F\in A(\omega_n)$, then $F(f(x))\in A$ if and only if f is real and $\|e^{inf}\|=O(\omega_n)$, as $n\to\pm\infty$. With the help of this result and a classical result of Wiener relating to the absolute convergence of Fourier series he deduces: If in the neighbourhood of every point x, f is equal to a real function f_x for which $\|e^{inf_x}\|=O(\omega_n)$ as $n\to\pm\infty$, then $\|e^{inf}\|=O(\omega_n)$. He uses these two results to prove a number of theorems concerning the rapidity of the growth of $\|e^{inf}\|$ when f is either linear by intervals or analytic. The following two results are typical: (1) If f be real, continuous and linear by intervals satisfying $f(x+2\pi)=f(x)$ (mod 2π), then $\|e^{inf}\|=O(\log n)$, $n\to\pm\infty$; moreover, if $f(x)=|x|$ in $[-\pi, \pi]$, then $\|e^{inf}\|=(2/\pi)\log n+O(1)$. (2) If f be a real, non-constant, analytic 2π-periodic function, then there exist two positive constants λ_1 and λ_2 such that $\lambda_1|n|^{\frac{1}{2}}<\|e^{inf}\|<\lambda_2|n|^{\frac{1}{2}}$ for all integral n. Several interesting corollaries are mentioned and finally an example f is given for which $f\in A^*$ but $|f|\notin A^*$.

U. N. Singh (Zbl **72**, 60)

4150:

Kahane, J. P. Sur un problème de Littlewood. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 268–271.

The author constructs a Fourier series $\sum_1^{\infty}\beta_n\sin nx$ with positive coefficients, such that the series $\sum_1^{\infty}\beta_n\cos nx$ is not a Fourier series, and thus solves one of Littlewood's problems. The reviewer has shown [#4148 above] that if $\sum_{-\infty}^{\infty}\phi(c_n)e^{inx}$ is a Fourier series whenever $\sum_{-\infty}^{\infty}c_n e^{inx}$ is a Fourier series, then the function ϕ satisfies a Lipschitz condition of order 1 in a neighborhood of the origin. The example of the present paper implies the existence of a Fourier series $\sum_{-\infty}^{\infty}c_n e^{inx}$ such that $\sum_{-\infty}^{\infty}|c_n|e^{inx}$ is not a Fourier series; hence the above-mentioned necessary condition is not sufficient. The construction of the ex-

ample uses a method which is analogous to one which the author has used earlier [#4149 above] to construct a function f whose Fourier series converges absolutely, whereas $|f|$ does not have this property.

W. Rudin (Rochester, N.Y.)

4151:

Kahane, Jean-Pierre. Sur un théorème de Wiener-Lévy. C. R. Acad. Sci. Paris **246** (1958), 1949–1951.

Désignons par A l'ensemble des fonctions à valeurs complexes, 2π-périodiques, dont la série de Fourier converge absolument. Soit A_r la partie de A formée de $f \in A$ à valeurs réelles. On sait, d'après Wiener et P. Lévy, que si F est une fonction analytique de variable réelle et si $f \in A_r$, $F \circ f$ $[F \circ f(x) = F(f(x))]$ appartient à A. Désignons par $K\{M_n\}$ la classe des fonctions F sur R indéfiniment dérivables, à valeurs complexes, telles que $|F^{(n)}(x)| \leq AM_n$ $(x \in R, n \geq 0)$. L'auteur démontre que si la classe $K\{M_n\}$ contient des fonctions non analytiques, il existe une $F \in K\{M_n\}$ et une $f \in A_r$ telles que $F \circ f \notin A$. {La classe $K\{M_n\}$ est définie, dans la note, d'une manière un peu plus générale, mais la démonstration ne semble d'appliquer qu'au cas cité.} L'auteur suppose qu'il doit être difficile de construire une fonction F non analytique telle que pour toute $f \in A_r$ on ait $F \circ f \in A$, et ceci en vertu de la proposition suivante (qu'il démontre): Si F est périodique et telle que quel que soit $f \in A_r$, $\|F \circ (f+a)\| \leq A < \infty$ (A indépendant de a, réel), F est analytique (si $\varphi(x) = \sum a_n e^{inx}$, $\varphi \in A$, $\|\varphi\| = \sum |a_n|$). Cette proposition va jouer un rôle important dans les travaux qui suivent la note de Kahane, et qui donnent une solution complète du problème: existe-t-il une fonction non analytique F telle que $F \circ f \in A$ pour toute $f \in A_r$?

S. Mandelbrojt (Paris)

4152:

Katznelson, Yitzhak. Sur les fonctions opérant sur l'algèbre des séries de Fourier absolument convergentes. C. R. Acad. Sci. Paris **247** (1958), 404–406.

Dans cette note Katznelson donne la réponse complète au problème cité dans la note de Kahane [voir l'analyse ci-dessus]. L'auteur démontre que pour que $F \circ f \in A$, pour toute $f \in A_r$, F étant 2π-périodique, il faut que F soit analytique (les notations sont les mêmes que dans la note de Kahane). D'une manière plus précise, l'auteur généralise d'abord le résultat de Kahane cité à la fin de l'analyse précédente. Soit F une fonction 2π-périodique; si pour tout f tel que $f \in A_r$, $\|f\| < R$ et tout a, avec $-\pi < a \leq \pi$, on a $F \circ (f+a) \in A_r$, $\|F \circ (f+a)\|$ étant borné (par une quantité dépendant seulement de R), F est analytique dans une bande de largeur $2R$ autour de l'axe réel. Suivant alors une idée de Helson, l'auteur démontre (en se basant sur le résultat précédent) le théorème précis que voici: Soit I un intervalle ouvert sur l'axe réel, soit $F(x)$ définie sur I telle que, pour tout $f \in A_r$ prenant ses valeurs dans I, $F \circ f \in A_r$; dans ces conditions $F(x)$ est analytique sur I.

S. Mandelbrojt (Paris)

4153:

O'Shea, Siobhan. Note on an integrability theorem for sine series. Quart. J. Math. Oxford Ser. (2) **8** (1957), 279–281.

Heywood [Quart. J. Math. Oxford Ser. (2) **5** (1954), 71–76; MR **16**, 30] showed that if $b_n \downarrow 0$ and $g(x) = \sum_{n=1}^{\infty} b_n \sin nx$, then $x^{-\gamma} g(x) \in L$ if and only if $\sum n^{\gamma-1} b_n < \infty$, where $1 < \gamma < 2$. The author now shows that the assumption $b_n \downarrow 0$ can be dropped altogether: the series for $g(x)$ converges to a function satisfying $x^{-\gamma} g(x) \in L$ if and only if $\sum n^{\gamma-1} b_n < \infty$. The extended theorem fails for $0 < \gamma \leq 1$, but the monotonicity of b_n can be replaced by the condition that $b_n \to 0$ and $n^{-k} b_n$ is nonincreasing for some nonnegative k.

R. P. Boas, Jr. (Evanston, Ill.)

4154:

Yamamoto, Nobuko. On E. Hille's theorem. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **7** (1957), 13–17.

The author proves that every bounded linear operator on $L^1(G)$ (where G is a locally compact abelian (l.c.a.) group) which commutes with all translations of G corresponds to multiplication of the Fourier transforms by a bounded function (the factor function of T), and furthermore, for discrete G, that this function is a Fourier-Stieltjes transform. {He seems to be unaware that this was proved, in the general case, by Helson [Ark. Math. **2** (1953), 475–487; MR **15**, 327].} For compact G, the author establishes the same conclusion for operators on $C(G)$, the space of all continuous functions on G {there seems to be a gap in step (a) of the proof (however, the theorem is correct)}. Finally, he considers weakly measurable semi-groups of such operators on $L^1(G)$, for any l.c.a. G, and discusses the properties of the corresponding factor functions.

W. Rudin (New Haven, Conn.)

4155:

Helson, Henry; and Lowdenslager, David. Prediction theory and Fourier series in several variables. Acta Math. **99** (1958), 165–202.

Let S be a set of pairs (m, n) of integers such that: $(0, 0) \notin S$; $(m, n) \in S$ if and only if $(-m, -n) \notin S$ (unless $m=0$, $n=0$); $(m, n) \in S$ and $(m', n') \in S$ imply $(m+m', n+n') \in S$. Consider the torus T, written as all pairs (x, y) of real numbers such that $0 \leq x < 2\pi$, $0 \leq y < 2\pi$, with addition in both components modulo 2π, and Haar measure $d\sigma = (4\pi^2)^{-1} dxdy$. The set S forms a subset of the character group of T, in a natural way. Let μ be a nonnegative finite measure on T with absolutely continuous component $w(x, y)d\sigma$. Then

$$\exp \int_T \log w \, d\sigma = \inf_P \int_T |1+P|^2 d\mu,$$

where P runs through all finite sums of the form $\sum_S a_{mn} e^{-i(mx+ny)}$. The left side is defined as zero if $\log w$ is not summable. This generalizes a theorem of G. Szegö [Math. Z. **6** (1920), 167–202; see also N. Ahiezer, Lectures on the theory of approximation, OGIZ, Moscow-Leningrad, 1947; MR **10**, 33; pp. 274–283 and the literature there cited]. A number of interesting consequences are drawn from this theorem. If $f \in L_1(T)$ and the Fourier coefficients of f vanish outside S, then $\int_T \log |f| d\sigma \geq \log |\int_T f d\sigma|$. If also $\int f d\sigma \neq 0$, then there are functions g and h in $L_2(T)$ whose Fourier coefficients also vanish outside S such that $f = gh$. Also a new proof is given of Bochner's generalization of a theorem of F. and M. Riesz [Ann. of Math. (2) **45** (1944), 708–722; MR **6**, 124]: if μ is a complex measure on T whose Fourier-Stieltjes coefficients vanish outside of a sector in the plane of angle $> \pi$ radians, then μ is absolutely continuous. The main theorems admit generalizations to arbitrary compact Abelian groups with ordered character groups, and also to matrix-valued functions and measures in place of complex-valued ones. The proofs are carefully given to apply to all cases. The connection of these results with certain facts of prediction theory is pointed out; see, in particular, Wiener and Masani [#4323, 4325 below].

E. Hewitt (Seattle, Wash.)

4156:

Cuculescu, Ion. Généralisation aux groupes quelconques d'un théorème de E. Hille concernant les fonctions facteurs. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 15–19.

Let G be a locally compact abelian group and $\hat{G}$ be its character group. Let φ be a complex-valued function on G with the following property: For each function h on G which is the Fourier transform of a summable function on $\hat{G}$, the product $\varphi \cdot h$ is the Fourier transform of a summable function on $\hat{G}$. Then φ is called a factor function. Hille showed in his book "Functional analysis and semi-groups" [Amer. Math. Soc. Colloq. Publ., vol. 31, New York, 1948; MR **9**, 594; Th. 18.2.2, pp, 362–363] that a function on the group of reals is a factor function if and only if it is the Fourier transform of a bounded measure. The author extends this theorem to the case of a general abelian locally compact group, with the bounded measure defined on $\hat{G}$. The same extension was made by H. Helson [Ark. Mat. **2** (1953), 475–487; MR **15**, 327; Theorem 1].

J. Wermer (Providence, R.I.)

INTEGRAL TRANSFORMS

See also 4018, 4136, 4186, 4409, 4410.

4157:

Delange, Hubert. Un théorème sur l'intégrale de Laplace-Stieltjes. Ann. Sci. École Norm. Sup. (3) **75** (1958), 1–17.

The paper supplies a detailed proof of a theorem on the singularity of the function $f(s)=\int_0^\infty e^{-st}d\alpha(t)$ announced in an earlier paper [Rend. Sem. Mat. Fis. Milano **26** (1957), 88–102; MR **19**, 413].

T. H. Hildebrandt (Ann Arbor, Mich.)

4158:

Gegelia, T. G. On properties of certain classes of continuous functions under a Hilbert transformation in E^n. Soobšč. Akad. Nauk Gruzin. SSR. **19** (1957), no. 3, 257–261. (Russian)

The author obtains certain inequalities for integrals similar to those considered earlier by him [same Soobšč. **16** (1955), 657–663; MR **17**, 953].

H. P. Thielman (Ames, Iowa)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 4192.

4159:

Widom, Harold. Equations of Wiener-Hopf type. Illinois J. Math. **2** (1958), 261–270.

The author studies the integral equation of the Wiener-Hopf type

$$(*)\qquad \int_0^\infty h(x-y)f(y)dy=f(x)$$

for the case in which $k(x)\in L_2(-\infty,\infty)$. This is to be contrasted to the original case of Wiener and Hopf in which $k(x)$ vanishes exponentially as $x\to\pm\infty$. {Reviewer's note: $k(x)$ need only vanish exponentially in one direction and may be exponentially large in the other, as long as the growth and decay constants satisfy an appropriate inequality. In the light of this remark and a recent paper by the reviewer [#4409 below], the remarks of the author regarding the work of Carlson and Heins [Quart. Appl. Math. **4** (1947), 313–329; MR **8**, 422] should be disregarded.} Using methods of analytic continuation developed by Carleman [L'integrale de Fourier et questions qui s'y rattachent, Publ. Sci. Inst. Mittag-Leffler, 1, Uppsala, 1944; MR **7**, 248], the author discusses a method of determining the solution of (*). The result depends on the zero of

$$(**)\qquad \int_{-\infty}^\infty e^{i\xi x}k(x)dx-1=0.$$

Sparenberg [Nederl. Akad. Wetensch. Proc. Ser. **A 59** (1956), 29–34; MR **17**, 976] considered the case in which the zeros were non-real, while the present author considers the more subtle case in which they are real.

A. E. Heins (Urbana, Ill.)

4160:

Položiĭ, G. N. A method of solving integral equations. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 876–878. (Russian)

In a manner analogous to one used in the theory of linear operators, the author considers in place of Fredholm's integral equations of the first and second kinds the single equation

$$\int_a^b K(x,s)\varphi(s)ds=\mu\varphi(x)+F(x).$$

He constructs a sequence of functions by the method of successive substitutions and states a number of theorems that give solutions of the integral equation, with symmetric kernels, and zero or non-zero values of μ, in terms of the limit of the constructed sequence of functions. The functions are assumed to be of the L^2-type. No proofs are given.

H. P. Thielman (Ames, Iowa)

4161:

Huang, Yuh-ren. A method of successive approximation of the Fredholm integral equation of the first kind. Advancement in Math. **3** (1957), 424–426. (Chinese. English summary)

Let $K(x,s)\in L_2(x,s)$ be a symmetric kernel. If the equation

$$(1)\qquad \int_a^b K(x,s)\varphi(s)ds=f(x),\quad f(x)\in L_2(a,b),$$

is solvable and if $\lambda^2<2\lambda_1^2$, where λ_1 is the characteristic value of the kernel $K(x,s)$ which is least in absolute value, then the solution of (1) is the limit in the mean of the sequence $\{\varphi_n(x)\}$ defined by

$$\varphi_n(x)=\nu_{n-1}(x)+\lambda[f(x)-f_{\nu_{n-1}}(x)],$$
$$\nu_{n-1}(x)=\varphi_{n-1}(x)-\lambda[f(x)-f_{\varphi_{n-1}}(x)],$$

where $\varphi_0(x)\in L_2(a,b)$ is an arbitrary function, and

$$f_{\nu_{n-1}}(x)=\int_a^b K(x,s)\nu_{n-1}(x)ds.$$

Author's summary

4162:

Yanovskiĭ, S. V. The relation of integral equations of convolution type to equations having a Cauchy kernel. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 458–461. (Russian)

Integral equations of the convolution type such as

$$f(x)+(2\pi)^{-\frac{1}{2}}\int_{-\infty}^\infty k(x,t)f(t)dt+\int_{-\infty}^\infty h(x,-t)f(t)dt=g(x),$$

where $k(x, t)$ is $k_1(x-t)$ or $k_2(x+t)$ depending on the sign of t, can be reduced by Fourier transforms to singular integral equations with Cauchy kernels. The author examines this reduction in detail and discusses the conditions imposed upon the kernels.

D. C. Kleinecke (Livermore, Calif.)

4163:

MacCamy, R. C. On singular integral equations with logarithmic or Cauchy kernels. J. Math. Mech. 7 (1958), 355–375.

The author translates integral equations such as

$$\int_0^2 [A(x, t)(x-t)^{-1}+B(x, t)\ln|x-t|+c(x, t)]f(t)dt=g(x)$$

into functional relations for multi-valued analytic functions of a complex variable. From this, even though the relations cannot be solved explicitly (in general) one can obtain information regarding the structure of the analytic function and this, in turn, can be related to facts concerning the integral equation. These function-theoretic notions date back to T. Carleman [Math. Z. **15** (1922), 111–120; Ark. Mat. Astr. Fys. **16** (1922), no. 26] for certain elementary integral equations. The necessary tools for a systematic investigation for a large class of integral equations were presented by Lewy [Proc. Internat. Cong. Math. 1950, vol. 1, pp. 601–605, Amer. Math. Soc., Providence, R.I., 1952; J. Math. Mech. **6** (1957), 91–107; MR **13**, 642; **18**, 883]. In particular, the author studies $f(t)$, $t\to 0$, under suitable restrictions on A and B.

A. E. Heins (Pittsburgh, Pa.)

4164:

Hellman, Olavi. On the periodicity of the solution of a certain nonlinear integral equation. Pacific J. Math. 8 (1958), 219–226.

Let $F(t)$ be a periodic function which is bounded and measurable on $[0, \infty]$ and $E(t)$ be a solution of the nonlinear integral equation,

$$E(t)=F(t)-\int_0^t G(t-\tau)N\{E(\tau)\}d\tau.$$

With suitable restrictions on F and G it is found that $\lim E(nT+u)=v(u)$ exists as $n\to\infty$ through integer values. Moreover, $v(u)$ is periodic with period T and is itself the solution of an auxiliary integral equation which can be solved by iteration.

R. C. MacCamy (Pittsburgh, Pa.)

4165:

Lin, Chün. A remark on the solution by mechanical quadrature of a non-linear integral equation. Advancement in Math. 4 (1958), 139–142. (Chinese)

4166:

Govoruhina, A. A. Integro-differential equations of convolution type. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 866–869. (Russian)

Leaning heavily on previous work by Gahov and Čerskiĭ [Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 33–52; MR **18**, 134] the author states, without proof, conditions sufficient for solving the integro-differential equation

$$(1)\quad \sum_{m=0}^{n}\Big[\lambda_m f^{(m)}(x)+(2\pi)^{-\frac{1}{2}}\int_0^\infty k_{1m}(x-t)f^{(m)}(t)dt+ (2\pi)^{-\frac{1}{2}}\int_0^\infty k_{2m}(x-t)\,f^{(m)}(t)dt\Big]=g(x)$$

in closed form by first transforming it into an equivalent Riemann boundary problem.

J. F. Heyda (Cincinnati, Ohio)

4167:

Melzak, Z. A. A scalar transport equation. II. Michigan Math. J. 4 (1957), 193–206.

[For part I, see Trans. Amer. Math. Soc. **85** (1957), 547–560; MR 19, 428]. The author proves the following theorem: Let $f(x, y)$ be a function for which $f(x, 0)$ is non-negative, bounded, continuous, integrable and satisfying the condition $\int_0^\infty xf(x, 0)dx<\infty$; let $\phi(x, y, t)$ be a function which is non-negative, bounded, continuous with respect to x, y, and t, with continuity in t uniform with respect to x and y, and with $\phi(x, y, t)=\phi(y, x, t)$; let $\psi(x, y, t)$ be a function which is non-negative bounded, continuous with respect to x, y, and t, with continuity in t uniform with respect to x, y; moreover, let $\psi(x, y, t)$ satisfy the conditions: $\int_0^x \psi(x, y, t)dy<E-1<\infty$; $x^{-1}\int_0^x y\psi(x, y)d, ty$ is bounded from above by 1 and continuous in x and t, and its continuity in t is uniform with respect to x. In that case, the equation

$$\partial f(x, t)/\partial t=\tfrac{1}{2}\int_0^x f(y, t)f(x-y, t)\phi(y, x-y, t)dy -f(x, t)\int_0^\infty f(y, t)\phi(x, y, t)dy+\int_0^\infty f(y, t)\psi(y, x, t)dy -[f(x, t)/x]\int_0^x y\psi(x, y, t)dy$$

possesses a solution $f(x, t)$ which is valid for $x, t\geqq 0$. This solution is continuous, non-negative, continuously differentiable in t for each x, and integrable in x for each t. It is the only solution of this equation which is continuous and integrable in x for all t and which also assumes the prescribed initial value $f(x, 0)$ for $t=0$.

D. ter Haar (Oxford)

FUNCTIONAL ANALYSIS

See also 3933, 3981, 4101, 4105, 4121, 4122, 4154, 4249, 4250.

4168:

Dynin, A. S. On spaces nuclear in different senses. Dokl. Akad. Nauk SSSR **121** (1958), 790–792. (Russian)

D. A. Raikov has proved [Uspehi Mat. Nauk (N.S.) **12** (1957), no. 5(77), 231–236; MR **19**, 967] that certain locally convex spaces are nuclear in the sense of Gelfand (Γ-nuclear) if they are nuclear in the sense of Grothendieck (G-nuclear). In the paper under review, the author sketches a proof that these two notions coincide for spaces of type (F) and complete spaces of type (DF). He outlines the construction of a complete G-nuclear space which is not Γ-nuclear.

R. R. Phelps (Princeton, N.J.)

4169:

Yoshinaga, Kyôichi. On a locally convex space introduced by J. S. E Silva. J. Sci. Hiroshima Univ. Ser. A **21** (1957/58), 89–98.

Le reviewer a considéré, sous le nom d'espace (LN^*), un certain type de limites inductives d'espaces normés qui se présente couramment dans les théories des distributions et des fonctionnelles analytiques [Rend. Mat. e Appl. (5) **14** (1955), 388–410; MR **16**, 1122]. L' A. étudie les mêmes espaces, en les appelant "espaces de Silva" et en les rattachant au travail de Grothendieck sur les espaces (F) et (DF) [Summa Brasil Math. **3** (1954), 57–122; MR **17**, 765]. L'A. donne d'abord plusieurs caractérisations vectoriel-topologiques de ces espaces: ils sont

identiques aux espaces (DF) de Montel (séparés) vérifiant la condition de Mackey stricte; leurs duals forts s'identifient aux espaces de Schwartz métrisables complets, etc. Ensuite l'A. démontre plusieurs propriétés de permanence de ces espaces et de leurs duals. Enfin, l'A. présente deux caractérisations intéressantes des espaces (LF) de Schwartz (ils sont identiques aux limites inductives strictes d'espaces (F) de Schwartz) et de leurs duals forts (ils sont identiques aux limites projectives de certains spectres d'espaces de Silva).

J. Sebastião e Silva (Lisbon)

4170:

Hirata, Yukio. On a theorem in an (LF) space. J. Sci. Hiroshima Univ. Ser. A **20** (1956/57), 111–113.

L'object essentiel de cet article est d'établir le théorème suivant: Si la limite inductive stricte d'une suite d'espaces métrisables est un espace distingué, le dual fort de cet espace est bornologique. Ce théorème géneralise un résultat de Grothendieck [Summa Brasil. Math. **3** (1954), 57–123; MR **17**, 765]; il est basé sur un lemme analogue à des résultats de Grothendieck et du reviewer, concernant la limite projective E d'une suite d'espaces E_n par rapport à des applications linéaires continues ϕ_n: si toute suite, convergeant vers 0 dans E [resp. toute partie bornée], est l'image d'une suite bornée [resp. partie bornée] de E, alors E est bornologique.

J. Sebastião e Silva (Lisbon)

4171:

Kolmogorov, A. N. On linear dimensionality of topological vector spaces. Dokl. Akad. Nauk SSSR **120** (1958), 239–241. (Russian)

Two definitions of dimension type are introduced, both stated for Fréchet spaces (type F). The first: $\delta(E) \leqq \delta(E')$ provided there is a continuous linear mapping of a subspace of E' onto E. Theorem 2: For reflexive Banach spaces, $\delta(E) \leqq \delta(E')$ if and only if the conjugate spaces satisfy the same inequality. Next, let $G, G_1, \cdots$, denote bounded domains in the complex plane of finite connectivity. Let A^G denote the space of functions analytic in G, with the usual compact-open topology. Then $\delta(A^G)$ is independent of G (Th. 3), but if $H = G_1 \times \cdots \times G_n$, the types $\delta(A^H) = \alpha_n$ satisfy $\alpha_n < \alpha_{n+1}$ (Th. 4).

The "approximate dimension" $da(E)$ of an F-space E is a set of real-valued functions on $(0, \infty)$, namely all those φ such that, for any compact K and open U in E, U containing zero, there is $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$, K can be covered by $[\varphi(\varepsilon)]$ translates of εU. For E^n, this is all those φ satisfying $\varepsilon^{-n} = o(\varphi)$ as $\varepsilon \to 0$ (Th. 6); for any infinite-dimensional Banach space, this is the empty set of functions (Th. 7). For A^H as defined above, H a product domain in n complex variables, $da(A^H)$ consists of those φ such that $(-\ln \varepsilon)^{n+1} = o(\ln \varphi)$ (Th. 9). The dimension is also determined explicitly for a space of infinitely differentiable functions on a torus.

J. Isbell (Seattle, Wash.)

4172:

Vainberg, M. M. Some questions of differential calculus in linear spaces. Advancement in Math. **4** (1958), 14–54. (Chinese)

A translation of the Russian article in Uspehi Mat. Nauk (N.S.) **7** (1952), no. 4(50), 55–102 [MR **14**, 384].

4173:

Kuller, R. G. Locally convex topological vector lattices and their representations. Michigan Math. J. **5** (1958), 83–90.

A locally convex vector lattice E is a vector lattice with a locally convex T_2 topology defined by a directed set $\{p_\alpha\}$ of semi-norms ($p_\alpha \leqq p_\beta$ if $p_\alpha(x) \leqq p_\beta(x)$ for all x), with the property that $|x| \leqq |y|$ implies $p_\alpha(x) \leqq p_\alpha(y)$ for all α. For each α, let E_α be the quotient space of E by the subspace $\{x : p_\alpha(x) = 0\}$ equipped with the norm induced by p_α, and let $E_\alpha\hat{}$ be its completion. Then $E_\alpha\hat{}$ is a Banach lattice, and if E is complete (as a linear topological space) then E is the projective limit of Banach lattices $\{E_\alpha\hat{}\}$ relative to the quotient maps $\pi_\alpha : E \to E_\alpha$. This paper treats representations of complete, locally convex vector lattices where the $\{E_\alpha\hat{}\}$ are already M or L spaces.

{Remark: There are some mistakes in the paper. For instance, on page 85, the space mentioned in (2) is not complete unless the underlying space is a k-space; the maps π_α are not necessarily open; and E_α need not be complete even when E is complete.}

I. Namioka (Ithaca, N.Y.)

4174:

Bonsall, F. F.; and Reuter, G. E. H. A fixed-point theorem for transition operators in an (L)-space. Quart. J. Math. Oxford Ser. (2) **7** (1956), 244–248.

Let V be an abstract (L) space and let V^+ denote the positive cone in V. If $\{x_n\}$ is a sequence of elements in V^+ such that $\{\|x_n\|\}$ is bounded, then $\lim\inf_{n\to\infty} x_n$ is defined as $\lim_{k\to\infty} y_k$, where $y_k = \inf_{k \leq n < \infty} x_n$. A transition operator in V is a bounded linear operator P in V satisfying the following conditions: (1) $Px \geqq 0$ ($x \in V^+$); and (2) $\|Px\| = \|x\|$. If instead of (2) P satisfies (2′) $\|Px\| \leqq \|x\|$, then P is called a contraction operator. Let $Q_n = (P + P^2 + P^3 + \cdots + P^n)/n$. The authors prove the following theorem: Let P be a contraction operator in an abstract (L) space V and let $u \in V^+$. Let $\{n_k\}$ be a strictly increasing sequence of integers. Then if $f = \lim\inf_{k\to\infty} Q_{n_k} u$ we have $Pf = f$.

Y. N. Dowker (London)

4175:

Mlak, W. Differential inequalities in linear spaces. Ann. Polon. Math. **5** (1958), 95–101.

In this paper forms of the generalized mean value theorem are stated and generalizations of classical differential inequalities based on partial orderings generated by convex cones are given.

Let E be a real topological vector space; let $\bar{E}$ be the set of linear continuous functionals on E. A function $x(t)$ from a real interval Δ to E is called weakly continuous if $\xi \in \bar{E}$ implies that the real-valued function $\xi x(t)$ is continuous on Δ. A typical generalized mean value theorem is stated as follows. Theorem 1. Suppose A to be a convex body in E and $x(t)$ to be weakly continuous in Δ. Suppose that for each $\xi \in \bar{E}$ there exists an at most denumerable set $Z(\xi) \subset \Delta$ such that for every $t \in \Delta - Z(\xi)$ there is a sequence $\{\tau_n\}$ of positive real numbers converging to zero and a sequence $y_n \in A$ such that $\lim_{n\to\infty} \xi\{(x(t+\tau_n) - x(t))/\tau_n - y_n\} = 0$. Then, for $t_1 \pm t_2$, t_1, $t_2 \in \Delta$, $(x(t_1) - x(t_2))/(t_1 - t_2) \in A$.

Now let E be a Banach space and let a function $f(t, x)$ be defined and continuous for $t_0 \leqq t \leqq t_0 + \alpha$ and $\|x - x_0\| \leqq r$, and take on values in a compact subset V of E. Assume moreover that if W is the convex hull of $V \cup \{\theta\}$, where θ is the origin of E, then the diameter of W is less than r/α. Then the following conclusion is proved: Theorem 6: There exists at least one solution, $y(t)$, of the differential equation $y' = f(t, y)$ defined on $t_0 \leqq t \leqq t_0 + \alpha$ and satisfying $y(t_0) = x_0$.

P. C. Hammer (Madison, Wis.)

4176:

Dzyadik, V. K. On a maximum problem in a normed linear space. Luc'kiĭ Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. 6 (1958), no. 3, 75–80. (Ukrainian)

In a Banach space B let there be given a finite or infinite sequence of elements $x_1, x_2, \cdots, x_n$, such that each element has a norm less than or equal to K. Furthermore, let $c_1, c_2, \cdots, c_n, \cdots$ be a sequence, finite or infinite, of non-negative numbers such that $\sum c_n$ converges. Next, consider all elements x of B which can be expressed in the form $x=\sum a_n x_n$, where the a_n are real numbers and are such that $|a_n|\leqq c_n$ for all n. The author establishes the following result:

$$\max_{|a_n|\leq c_n} \|\sum a_n x_n\| = \max_{|a|=c_n} \|\sum a_n x_n\|.$$

The proof is direct, elementary, and is based on the obvious inequality

$$\|x\|\leqq \max\{\|x-y\|, \|x+y\|\}.$$

H. P. Thielman (Ames, Iowa)

4177:

Kwan, Chao-chih. Sur le théorème du graphe fermé. Advancement in Math. 3 (1957), 670–672. (Chinese. French summary)

On montre que le théorème de résonance énoncé pour l'espace de Banach est une conséquence immédiate du théorème du graphe fermé en considérant un espace produit convenablement construit. *Author's summary*

4178:

Lau, Leung-sum. On a characterization of Hilbert spaces. Advancement in Math. 4 (1958), 462–464. (Chinese. English summary)

The following theorems are proved.

Theorem 1. Let B be a complex Banach space; the necessary and sufficient condition for B to be a complex Hilbert space is 1) $B\simeq B^*$ (B is conjugate isomorphic to B^*), 2) $f_x(x)=\|x\|^2$ for every x in B, where f_x is the corresponding functional of x under the isomorphism.

Theorem 2. Let B be a real Banach space; the necessary and sufficient condition for B to be a real Hilbert space is 1) $B=B^*$ (B is self-adjoint), 2) $f_x(x)=\|x\|^2$ for every x in B, where f_x is the corresponding functional of x under the isomorphism. *From the author's summary*

4179:

Kaz'min, Yu. A. Some remarks on completeness in E_1 and L_2. Uspehi Mat. Nauk (N.S.) 13 (1958), no. 3(81), 197–203. (Russian)

Let E_1 be the set of all functions analytic in the open unit disk $|z|<1$ and let H_2 be the subset of E_1 containing those functions f for which $H_2(f)=\lim_{\rho\to 1-}\int_0^{2\pi} f(\rho e^{i\theta})d\theta<\infty$. A sequence $\{f_n\}$ of elements of H_2 is called a basis in H_2 if each f in H_2 is the sum of a unique series $\sum_n a_n f_n(z)$ uniformly convergent in the interior of $|z|<1$. Letting $f(e^{i\theta})$ be the boundary value of f, when it exists, the author is concerned with completeness of the set $\{\operatorname{Re} f_n(e^{i\theta}), \operatorname{Im} f_n(e^{i\theta})\}$ in the real Hilbert space $L^2(0, 2\pi)$ when $\{f_n\}$ is a basis in H_2. *M. M. Day* (Urbana, Ill.)

4180:

Quigley, Frank. Approximation by algebras of functions. Math. Ann. 135 (1958), 81–92.

Let X be a locally compact Hausdorff space and $C(X)$ be the algebra of all continuous complex valued functions on X. Let B be a subalgebra of $C(X)$ which contains the constant functions and separates points of X. Let $\bar{B}_S$, for a compact subset S of X, be the uniform closure of the restrictions to S of functions in B. The first three sections of the paper contain a classification of various types of maximal ideals and some relationships between the maximal ideals of the algebras involved. This material is largely preliminary to the problem of what sort of question can be asked about $\bar{B}_S$. Several appropriate questions are formulated and there are many examples illustrating the ideas involved. Sections five and six are devoted to a notion of convexity in the situation above which generalizes that for analytic functions of several complex variables. Section seven contains some ideal-theoretic properties of certain rings of quotients. Section eight derives some topological properties of the ranges of functions in a closed subalgebra of $C(X)$, X compact. The final section extends the earlier work of Helson and Quigley on maximal closed subalgebras of $C(X)$, X compact [Proc. Amer. Math. Soc. 8 (1957), 111–114; MR 18, 911]. *H. S. Bear* (Seattle, Wash.)

4181:

Garber, E. D. On the complete continuity of an imbedding operator. Uspehi Mat. Nauk (N.S.) 13 (1958), no. 2(80), 169–173. (Russian)

Let Ω be an n-dimensional, finitely connected region with piecewise smooth boundary. A function φ defined on Ω is in $W_p^{(l)}$ if it has continuous derivatives of order including l and if these are pth-power summable. By proving two inequalities the author shows that a set bounded in $W_p^{(l)}$ is an equiuniformly continuous set of functions on Ω; this shows that the identity map of $W_p^{(l)}$ into $C(\Omega)$ is completely continuous.

M. M. Day (Urbana, Ill.)

4182:

✱Гельфанд, И. М.; и Шилов, Г. Е. Обобщенные функции и действия над ними. [Gel'fand, I. M.; and Šilov, G. E. Generalized functions and operations on them.] Obobščennye funkcii, Vypusk 1. [Generalized functions, part 1.] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 440 pp. 13.75 rubles.

Voici un aperçu du contenu du tome I de cet ouvrage, qui comprendra quatre tomes (les trois premiers par Gel'fand et Šilov, le quatrième par Gel'fand et Vilenkin).

Chap. I. Définition et propriétés simples des fonctions généralisées. Les auteurs désignent par K l'espace des fonctions indéfiniment différentiables à support compact dans R^n; convergence des suites (dans ce tome I on ne munit pas K de la topologie de Schwartz [Théorie des distributions, tomes I et II, Hermann, Paris, 1950, 1951; MR 12, 31, 833]; le tome II définira cette topologie). Définition usuelle des fonctions généralisées (ou distributions) comme formes linéaires continues (pour les suites) sur K; les auteurs adoptent une notation fonctionnelle: $T(x)$ désigne une distribution. Les propriétés simples sont données; les auteurs insistent sur le changement de variables dans les distributions. Autres exemples d'espaces fondamentaux (test spaces). Dans le § 3 les auteurs étudient de façon détaillée et avec de très nombreux exemples (p. 62–132) les parties finies. Le produit de composition est étudié au § 4.

Chap. II. Transformation de Fourier. On commence par définir la transformation de Fourier sur K; soit Z l'espace image. Par transposition on a la transformation de Fourier de Z' (espace dual de Z) dans l'espace des distributions. Exemples standard, et d'autres: en particulier,

les transformées de Fourier des parties finies étudiées au Chap. I, notamment de $(ax^2+bx+c)_+^\lambda$. Applications aux équations aux dérivées partielles; seuls quelques exemples très simples sont donnés; mais les auteurs consacreront le tome 3 à cette question. {Remarque du rapporteur: la formule d'inversion de Fourier apparait p. 175 sans signaler qu'il s'agit là d'un théorème. Au reste, on peut donner, selon L. Schwartz, une démonstration très simple de la formule d'inversion, en utilisant la solution de $xT=0$.}

Chap. III. Distributions sur une surface et solutions fondamentales des équations différentielles. On considère la variété S: $P(x)=0$, où P est une fonction indéfiniment différentiable, grad $P(x)$ n'étant jamais nul sur S. On désigne par ω (Leray) une forme différentielle d'ordre $n-1$ telle que $d(P\omega)=dx_1\cdots dx_n$; cette forme ω est définie de façon unique sur S. Définition de $\delta(P)$: $(\delta(P),\varphi)=\int_S\varphi(x)\omega$, $\varphi\in K(R^n)=K$. Exemples. Les auteurs introduisent ensuite les formes suivantes (Gel'fand et Šapiro): φ étant donné dans K, $\omega_0(\varphi)=\varphi\omega$; $d\omega_0(\varphi)=dP\cdot\omega_1(\varphi)$, $\cdots$, $d\omega_{k-1}(\varphi)=dP\cdot\omega_k(\varphi)$, $\cdots$; il existe des formes $\omega_k(\varphi)$ vérifiant ces relations; elle ne sont pas définies de façon unique sur S, mais $\int_S\omega_k(\varphi)$ est unique. Ceci justifie la définition de $\delta^{(k)}(P)$ par $(\delta^{(k)}(P),\varphi)=(-1)^k\int_S\omega_k(\varphi)$, $k=0, 1, \cdots$. Exemples. Propriétés diverses de $\delta^{(k)}(P)$, par exemple: $P\delta^{(k)}(P)+k\delta^{(k-1)}(P)=0$. Applications à $\delta^{(k)}(r^2-t^2)$. Généralisation au cas où S est une variété définie par k équations $P_i(x)=0$, les P_i étant indéfiniment différentiables et "se coupant bien".

Dans le § 2, les auteurs étudient les solutions fondamentales (élémentaires) d'équations aux dérivées partielles à coefficients constants, par la méthode de Herglotz. Exemples. {Remarque du rapporteur: Ce § devrait contenir le théorème de Malgrange et Ehrenpreis sur l'existence d'une solution élémentaire.}

Chap. IV. Fonctions généralisées homogènes. Le § 1 reprend, toujours avec de nombreux exemples, le travail de Gel'fand et Šapiro [Uspehi Mat. Nauk **10** (1955), no. 3(65), p. 3–70; MR **17**, 371; Amer. Math. Soc. Transl. (2) **8** (1958), 21–85]. Dans le § 2, on utilise le Chap. III, § 1, avec $P(x)=x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2$, $p+q=n$, de sorte que l'origine est cette fois point singulier sur S. Définition de $\delta^{(k)}(P)$ (p. 337) par des intégrales usuelles si $k<\frac{1}{2}((p+q)-2)$, des parties finies sinon. Définition des P_+^λ pour tout λ complexe: généralisation du classique travail de M. Riesz. Application à la recherche de solutions élémentaires pour L^k, où $L=\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_p^2-\partial^2/\partial x_{p+1}^2-\cdots-\partial^2/\partial x_{p+q}^2$. Calcul des transformées de Fourier des parties finies correspondants (p. 379–406) {les auteurs ne sont apparemment pas au courant des travaux de Méthée et de Rham sur ce sujet}. Le § 3 reprend sans changement une partie du travail de Gel'fand et Šapiro, loc. cit.

Ce tome I a un double but: (1) donner une exposition de la théorie des distributions de L. Schwartz (les auteurs ne se dirigent pas du tout vers les espaces de Soboleff et leurs nombreuses applications); dans ce tome I l'exposé est résolument élémentaire, toutes les questions de topologie étant laissées de coté; grâce aux très nombreux et intéressants exemples traités en détail, ce tome constitue un excellent livre d'exercises (sans aucun sens péjoratif!) à lire en même temps que le livre de L. Schwartz; (2) développer des applications et des compléments à la théorie des distributions, d'après les travaux de Gel'fand et Šilov, Gel'fand et Šapiro, Gel'fand et Graev, Vilenkin et d'autres; les chap. III et IV du présent tome entrent dans ce cadre.
J. L. Lions (Nancy)

4183:

Fung, Kang. On the duality of the spaces of distributions. Advancement in Math. **3** (1957), 201–208. (Chinese)

This paper discusses the reflexivity of the following topological vector spaces: the space ($\mathfrak{D}$) of complex functions on R^n of class C^∞ with compact support, the space ($\mathfrak{E}$) of functions of class C^∞ with arbitrary support, and the space ($\mathfrak{S}$) of functions φ of class C^∞ satisfying $\lim_{|x|\to\infty}|x^kD^p\varphi(x)|=0$ for any integer $k\geq 0$ and any system $p=\{p_1, p_2, \cdots, p_n\}$ of n non-negative integers. The reflexivity of these spaces was first proved by L. Schwartz [Théorie des distributions, I, II, Hermann, Paris, 1950, 1951; MR **12**, 31, 833].
Ky Fan (Notre Dame, Ind.)

4184:

Bouix, Maurice. Application des distributions aux équations de Maxwell et de Helmholtz. C. R. Acad. Sci. Paris **246** (1958), 2858–2860.

The author uses the Dirac delta function for a surface S in order to express the boundary conditions for S derived from Maxwell's equations for the electric and magnetic potentials. Application is made to the current system when S is a sphere.
G. Temple (Oxford)

4185:

Šapiro, Z. Ya. On a class of generalized functions. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 3(81), 205–212. (Russian)

The k-fold δ-function in euclidean n-space is defined as the product of the δ-functions in each of the first k coordinates.

The author shows that in case of curvilinear coordinates a δ-function depending on the first k curvilinear coordinates can be similarly defined.
W. T. van Est (Utrecht)

4186:

Ishihara, Tadashige. Divergent integrals as viewed from the theory of functional analysis. II. Proc. Japan Acad. 33 (1957), 124–127.

This is a continuation of part I [same Proc. **33** (1957), 92–97; MR **19**, 755]; we use the terminology of that review. The author gives several examples of his generalized Laplace transform and shows that there is a $v^*\in\Phi'$ for which $\partial v^*/\partial\bar{k}=0$ but v^* is not an analytic function.
L. Ehrenpreis (Waltham, Mass.)

4187:

Fuglede, Bent. Extremal length and functional completion. Acta Math. **98** (1957), 171–219.

In the reviewer's opinion the best way of reporting the contents of this paper is in the terminology of functional spaces. [See N. Aronszajn and K. T. Smith, Ann. Inst. Fourier, **6** (1955–1956), 125–185; MR **18**, 319.] The author considers essentially two kinds of functional spaces, and his main results concern the relationships between the exceptional sets of these spaces. First kind: The basic set is the class $\mathscr{M}$ of all Borel measures in a Euclidean $\mathscr{R}^n$. The functional space $\mathscr{F}_p$, $p\geq 1$, is formed as follows. To each Baire function f, $f\in L^p(\mathscr{R}^n)$, we assign the function $u(\mu)$: $\mu\in\mathscr{M}$, $u(\mu)=\int f d\mu$ when f is integrable with respect to μ, otherwise $u(\mu)$ is undefined. The functions u form a pseudofunctional linear class whose exceptional sets $\mathscr{E}\subset\mathscr{M}$ are characterized by the property that there exists a Baire function $g\in L^p$, $g\geq 0$, such that $\int g d\mu=+\infty$ for all $\mu\in\mathscr{E}$. If we then put $\|u\|=$ the L^p-norm of the corresponding f, $\mathscr{F}_p$ becomes a

complete pseudo-functional space. The theorems 1 to 3 pertain essentially to this character of the space $\mathscr{F}_p$. Among special features of the space $\mathscr{F}_p$, the following should be noted: for $\mathscr{E} \subset \mathscr{M}$, put $M_p(\mathscr{E}) = \inf \|u\|^p$ for all $u \in \mathscr{F}_p$ with $u(\mu) \geqq 1$ for $\mu \in \mathscr{E}$; then $M_p(\mathscr{E})$ is a σ-subadditive set function, and $M_p(\mathscr{E})=0$ if and only if $\mathscr{E}$ is exceptional (since $\mathscr{F}_p$ has a strong majoration property, the general theory gives that $\delta(\mathscr{E}) = M_p(\mathscr{E})^{1/p}$ is σ-subadditive; the stronger property here is due to the connection between $\mathscr{F}_p$ and L^p). In particular, the author considers measures which are the k-dimensional areas on k-dimensional Lipschitz surfaces (these measures are identified with the corresponding surfaces). When the space is a set of rectifiable arcs ($k=1$) in the plane ($n=2$) and $p=2$, $1/M_p$ coincides with the extremal length of the class $\mathscr{E}$ as introduced by Ahlfors and Beurling — whence the first part of the title.

Second kind: The basic set is R^n. The functional space $P_{\alpha,p}$, $0<\alpha<n$, $p \geqq 1$, is the space of all potentials $u(x) = \int_{R^n} f(y)|x-y|^{\alpha-n}dy$, for $f \in L^p$. We put $\|u\| = [\int |f|^p dx]^{1/p} + \int_{|x|>1} |x|^{\alpha-n}|f|dx$. Thus, $P_{\alpha,p}$ becomes a complete space with exceptional sets $\mathscr{E}$ characterized by the property that there exists a $g \in L^p$ with $\int_{|x|>1} |x|^{\alpha-n}|g|dx < \infty$ such that $g \geqq 0$ and $\int g(y)|x-y|^{\alpha-n}dy = +\infty$ for $x \in E$. For integral α the space $P_{\alpha,p}$ is essentially the perfect functional completion of the class $C^{(\alpha)}$ provided with a norm whose principal part is the sum of L^p-norms of all the derivatives of order α — whence the second part of the title. A Theorem is stated which relates the exceptional sets of $P_{\alpha,p}$ to those of $P_{\alpha p/2,2}$ and those of $P_{(\alpha p/2)\pm\epsilon,2}$ ($\pm$ depending on $p<$ or >2). This general theorem will be proved in a forthcoming paper by the author; particular cases of it were obtained by Littlewood and Duplessis, J. Deny and H. Cartan. The main theorem of the paper (theorem 6) is the following: for $E \subset \mathscr{R}^n$, let $S^k(E)$ be the class of all k-dimensional Lipschitz surfaces intersecting E and $\mathscr{E}^k \subset \mathscr{M}$ the set of corresponding measures. Further, let $p>1$ and $kp \leqq n$. Then E is exceptional for $P_{k,p}$ if and only if $\mathscr{E}^k$ is exceptional for $\mathscr{F}_p$. For $p=1$, a somewhat less precise relation is obtained. For $p=2$, theorem 9 states: if $K \subset R^n$ is compact, C=the class of all Lipschitz arcs joining K to ∞, and H=the class of all Lipschitz $(n-1)$-surfaces separating K from ∞, then $M_2(C) = M_2(H)^{-1} = a_n \operatorname{cap}_2 K$, with a_n constant. This is an extension of a known formula in the theory of extremal length. Among applications to functional completion obtained by the author, we mention here the following: an L^p-irrotational vector-field in a region $X \subset R^n$ is defined as an L^p limit of smooth irrotational vector fields in X. Such a vector field F is characterized by the facts that it is L^p in X and that for all closed rectifiable curves in X homologous to 0 in X, except for a class of such curves exceptional for $\mathscr{F}_p$, the integral of F is 0.

N. Aronszajn (Lawrence, Kans.)

4188:

Stampacchia, Guido. Completamenti funzionali ed applicazione alla teoria dei potenziali di dominio. Rend. Mat. e Appl. (5) **16** (1957), 415–429.

Exposé de la théorie de la complétion fonctionnelle selon Aronszajn et Smith [Ann. Inst. Fourier **6** (1955–56), 125–185; MR **18**, 319]. Application à diverses normes définies par des opérateurs différentiels, par exemple $\|f\| = \|f\|_p + \|D_1 f\|_p$, où f est continuement dérivable dans un cube T de R^n, et $\|\cdot\|_p$ désigne la norme dans $L_p(T)$; si $1<p<n$ on prend pour classe exceptionnelle la classe des ensembles dont les projections sur les plans à $n-[p]$ dimensions sont de mesure nulle; la complétion fonctionnelle correspondante n'est pas parfaite au sens de Aronszajn et Smith [loc. cit.]. *J. Deny* (Strasbourg)

4189:

Lau, Leung-sum. A note on the spectral decomposition of self-adjoint operators. Advancement in Math. **4** (1958), 277–281. (Chinese)

It is well known [K. Friedrichs, Math. Ann. **109** (1934), 465–487] that the spectral resolution for a half-bounded self-adjoint operator can be easily derived from that for a bounded self-adjoint operator. In the present paper the classical spectral resolution theorem for unbounded self-adjoint operators is proved by reducing the general case to the half-bounded case. *Ky Fan* (Notre Dame, Ind.)

4190:

Putnam, C. R.; and Wintner, A. On the spectra of group commutators. Proc. Amer. Math. Soc. **9** (1958), 360–362.

The authors prove the following theorem: Let A, B be bounded invertible operators on a Hilbert space and suppose that A commutes with $AB-BA$; suppose further that A has a logarithm which commutes with all operators which commute with A. Then the spectrum of $ABA^{-1}B^{-1}$ consists only of 1. In case A is completely continuous, the same result follows without assumptions on the logarithm of A. *I. N. Herstein* (Ithaca, N.Y.)

4191:

Allen, H. S. The intersection of the Köthe-Toeplitz maximal matrix rings. Quart. J. Math. Oxford Ser. (2) **7** (1956), 277–279.

Let $\mathfrak{B}$ be the set of all infinite matrices with only finitely many elements different from zero; let $\mathfrak{D}$ be the set of all diagonal matrices. The author proves that a matrix A maps every "convergence-free" perfect Köthe-Toeplitz sequence space into itself if and only if $A = B+D$, $B \in \mathfrak{B}$, $D \in \mathfrak{D}$. A maps every perfect Köthe-Toeplitz space into itself if and only if $A=B+D$, and the elements of D are bounded. Let $\sigma(\alpha)$ denote the ring of matrices which map a perfect Köthe-Toeplitz space α into itself. The author also identifies matrices which belong to $\sigma(\alpha)$ for each α and have a two-sided reciprocal in $\sigma(\alpha)$.

G. G. Lorentz (Syracuse, N.Y.)

4192:

Sikorski, R. On determinants of Leżański and Ruston. Studia Math. **16** (1957), 99–112.

The author compares the generalization of the Fredholm theory of integral equations developed by the reviewer [Proc. London Math. Soc. (2) **53** (1951), 109–124; (3) **1** (1951), 327–384; MR **13**, 138, 468] for operators of the trace class with that developed by T. Leżański [Studia Math. **13** (1953), 244–276; **14** (1953), 13–23; MR **15**, 535, 881]. He shows that Leżański's theory can be applied to every operator of the trace class, and gives examples of operators outside the trace class (but necessarily asymptotically quasi-compact [J. London Math. Soc. **29** (1954), 318–326; MR **15**, 965]) to which his theory can also be applied.

{Notes: The reviewer finds himself unable to accept the author's proof (p. 101) that $\|F_{K_0}\| = \|K_0\|^*$, except when X is reflexive. Footnote 4 (p. 106) appears to be at variance with the text on that page.}

A. F. Ruston (Sheffield)

4193:

Glazman, I. M. On expansibility in a system of eigenelements of dissipative operators. Uspehi Mat. Nauk (N.S.) 13 (1958), no. 3(81), 179–181. (Russian)

Let A be a linear bounded operator on Hilbert space such that (1) the imaginary part of (Af, f) is non-negative for all f, and (2) A has a system $\{\varphi_k\}$ of normalized characteristic vectors with corresponding characteristic numbers λ_k such that

$$\sum_{j\neq k} (\mathrm{Im}\,\lambda_j)(\mathrm{Im}\,\lambda_k)/|\lambda_j-\lambda_k|<\infty;$$

then $\{\varphi_k\}$ is a "Riesz basis" for its closed linear hull; that is, there exist positive real numbers m and M such that if $f=\sum_k c_k\varphi_k$, then

$$m\sum_k |c_k|^2 \leqq \|f\|^2 \leqq M\sum_k |c_k|^2.$$

M. M. Day (Urbana, Ill.)

4194:

Sobolevskiĭ, P. E. On equations with operators forming an acute angle. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 754–757.

Soit A et B deux opérateurs non bornés dans une espace de Hilbert H, avec même domaine D de définition. On dit que A et B font un angle aigu si (a) $(Ax, Bx)\geqq m\|Ax\|\,\|Bx\|$ pour tout $x\in D$, $0<m\leqq 1$. Diverses propriétés sont données; par exemple, A et B ont mêmes indices de défaut. Supposons A et B auto adjoints >0, avec (a), et $\alpha_1\|Ax\|\leqq \|Bx\|\leqq\alpha_2\|Ax\|$ pour tout $x\in D$, α_1, $\alpha_2>0$. On donne par ailleurs un opérateur F non borné, de domaine $\supset D$, avec $\|Fx\|^2\leqq\delta^2\|Ax\|^2+c^2\|A^{1-\varepsilon}x\|^2$, $x\in D$, pour c et $\varepsilon>0$ convenables, $0<\delta<\alpha_1 m$. Alors $B+F+kI$ (I=identité) fait un angle aigu avec A pour k assez grand.

Applications aux équations $du/dt+Au=f$ (ou $du/dt+A(t)u=f$). Exemples d'opérateurs différentiels elliptiques du deuxième ordre faisant un angle aigu.

J. L. Lions (Nancy)

4195:

Foiaş, Ciprian. Sur la décomposition spectrale en opérateurs propres des opérateurs formellement symétriques. C. R. Acad. Sci. Paris **246** (1958), 3147–3149.

Let $\mathscr{E}$ be a nuclear space with $\mathscr{E}^*$ the space of linear forms on $\mathscr{E}$ continuous in the topology of uniform convergence on bounded sets of $\mathscr{E}$ and such that $\langle x^*, \lambda x\rangle=\bar\lambda\langle x^*, x\rangle$. Let $(x|y)$ be a hermitian form on E such that $\|x\|=\sqrt{(x|x)}$ is a continuous norm on $\mathscr{E}$. Let $\mathscr{H}$ be the Hilbert space formed by completing $\mathscr{E}$ with respect to this norm. Let A be a continuous linear operator on E such that $(Ax|y)=(x|Ay)$. Since $\mathscr{E}\subset\mathscr{H}\subset\mathscr{E}^*$, the adjoint A^* of A can be considered an extension of A. Let $\|x\|_1\geqq\|x\|$ be the semi-norm of the family of semi-norms defining the topology of $\mathscr{E}$, such that the embedding of $\mathscr{E}_1$ in $\mathscr{H}$ is a nuclear transformation, where $\mathscr{E}_1$ is the completion of $\mathscr{E}$ with respect to $\|x\|_1$. The following theorem is stated. To each maximal symmetric extension $\hat{A}$ of A in H there corresponds a positive bounded measure $\sigma(\lambda)$ on $(-\infty, \infty)$ unique to sets of measure 0 and a family of operators χ_λ defined almost everywhere with respect to σ, continuous on $\mathscr{E}_1$ with values in $\mathscr{E}_1^*$ such that: (1) $\langle\chi_\lambda x, y\rangle$ is σ-integrable for all $x, y\in\mathscr{E}_1$ and the measure $\langle\chi_\lambda x, y\rangle d\sigma(\lambda)$ is uniquely defined; (2) $(x|y)=\int_{-\infty}^{\infty}\langle\chi_\lambda x, y\rangle d\sigma(\lambda)$ for all $x, y\in\mathscr{E}_1$; (3) for $x, y\in\mathscr{E}_1$, $\langle\chi_\lambda x, y\rangle=\langle\chi_\lambda y, x\rangle$; (4) for $x, y\in\mathscr{E}$, $(Ax|y)=\int_{-\infty}^{\infty}\lambda\langle\chi_\lambda x, y\rangle d\sigma(\lambda)$; (5) $A\chi_\lambda=\chi_\lambda A=\lambda\chi_\lambda$.

An application of this type of spectral decomposition is given to the case of a differentiable operator defined on the space of infinitely differentiable functions on R^n.

R. E. Fullerton (College Park, Md.)

4196:

Porath, Günter. Störungstheorie für abgeschlossene lineare Transformationen im Banachschen Raum. Math. Nachr. **17** (1958), 62–72.

Let T be a closed linear operator with domain D and range R in the complex Banach space B. Let $\{T_n\}$ be a sequence of linear operators, each with domain D, and each having range in B. Suppose $\{\varepsilon_n\}$ is a sequence of positive constants such that $\varepsilon_n\to 0$ and

$$\|(T_n-T)f\|\leqq\varepsilon_n(\|f\|+\|Tf\|)$$

for each $f\in D$. Then T_n is closed when n is sufficiently large. Furthermore, if K is a compact subset of the resolvent set $\rho(T)$, then $K\subset\varphi(T_n)$ for sufficiently large n, and the resolvent operator $(z-T_n)^{-1}$ converges to $(z-T)^{-1}$ uniformly in K, in the operator-norm topology.

Continuing with the same assumptions, let σ be a bounded component of the spectrum $\sigma(T)$, and let σ be inside a closed rectifiable Jordan curve C which separates it from the rest of $\sigma(T)$. Then, for sufficiently large n, C lies in $\rho(T_n)$, and hence there is a well-determined part of $\sigma(T_n)$ which lies inside of C; call it σ_n. Let P and P_n be the spectral projections associated with T, σ and T_n, σ_n, respectively. Then $\|P_n-P\|$ and $\|T_nP_n-TP\|$ are both $O(\varepsilon_n)$. In particular, if σ consists of a single point z_0 and if $\eta>0$, then, for n sufficiently large, $z\in\sigma_n$ implies $|z-z_0|<\eta$. In this case, if P has finite-dimensional range, of dimension m, then σ_n consists of at most m points each an eigenvalue of T_n, and the range of P_n is also m-dimensional. If $m=1$, then σ_n consists of a single eigenvalue z_n of T_n.

An example shows, however, that the following can occur: σ is composed of a single eigenvalue, but σ_n contains no eigenvalue. In the example, of course, P does not have a finite-dimensional range. In fact, $P=I$.

The author apparently was not acquainted with the work of J. D. Newburgh on the variation of spectra [Duke Math. J. 18 (1951), 165–176; MR 14, 481]. A number of the results here mentioned were proved by Newburgh. Moreover, Newburgh was less restrictive about the domains of the operators T_n.

A. E. Taylor (Los Angeles, Calif.)

4197:

Mikusiński, Jan G.-. Sur le fondement de calcul opératoire. Advancement in Math. 3 (1957), 375–395. (Chinese)

A translation of the French original in Studia Math. 11 (1949), 41–70 [MR 12, 189].

4198:

Loo, Win. Sur la méthode de Kantorovič. Advancement in Math. 2 (1956), 711–713. (Chinese)

The method referred to, for approximating a nonlinear operator in a Banach space, appeared in Trudy Mat. Inst. Steklov. 28 (1949), 104–144 [MR 12, 419].

4199:

Kantorovič, L. V. Approximate solution of functional equations. Advancement in Math. 4 (1958), 1–13. (Chinese)

A translation of the Russian article in Uspehi Mat. Nauk (N.S.) 11 (1956), no. 6(72), 99–116 [MR 18, 747].

CALCULUS OF VARIATIONS

4200:

Klötzler, Rolf. Bemerkungen zu einigen Untersuchungen von M. I. Višik im Hinblick auf die Variationsrechnung mehrfacher Integrale. Math. Nachr. **17** (1958), 47–56.

The author applies Višik's theory of strongly elliptic differential operators [Mat. Sb. N.S. **29(71)** (1951), 615–676; MR **14**, 171] to the problem of minimizing a general variational integral for n-fold non-parametric surfaces $u(x)$ in N-space, $x=(x_1, \cdots, x_n)$, $u=(u_1, \cdots, u_N)$, where the integrand depends on x, u, and the partial derivatives of u of orders 1, $\cdots$, m. He defines a Hermitian differential operator Lu of order $2m$ such that the auxiliary minimum problem becomes the minimization of $\delta^2I=[Lu, u]$ subject to zero boundary data for u and its derivatives of orders 1, $\cdots$, $m-1$. Let λ_1 denote the least eigenvalue. Then $\lambda_1 \geqq 0$ is necessary for a weak minimum, and $\lambda_1<0$ is sufficient.

W. H. Fleming (Providence, R.I.)

4201:

Baiada, E.; e Tripiciano, G. Un integrale analogo a quello di Weierstrass nel calcolo delle variazioni in una variabile. Rend. Circ. Mat. Palermo (2) **6** (1957), 263–270.

This note shows that a calculus of variations integral, taken over a curve $x=x(t)$, $y=y(t)$ $(t_0 \leqq t \leqq t_1)$ of finite length, which is given by the Lebesgue integral $\int_{t_0}^{t_1} F(x, y, x', y')dt$ when the functions $x(t)$ and $y(t)$ are absolutely continuous, is expressible as

$$\lim_{h\to 0}\int_{t_1}^{t_0-h} F[x(t), y(t), (x(t+h)-x(t))/h, (y(t+h)-y(t))/h]dt,$$

without the need for any restriction on the parametric representation of the curve.

L. M. Graves (Chicago, Ill.)

4202:

Kerimov, M. K., The Bliss condition for discontinuous variational problems with movable end-points. Azerbaĭdžan. Gos. Univ. Uč. Zap. **5** (1958), 17–23. (Russian. Azerbaidzani summary)

Let L_1, L_0, L_2 be three given simple unbounded continuous curves of the (x, y)-plane dividing the plane into four unbounded parts D_0, D_1^-, D_1^+, D_2. Let $\{C\}$ be the family of all continuous curves which are sums of two arcs C^-, C^+, $C^-\subset D_1^-$, $C^+\subset D_1^+$, where C^- joins L_1 to L_0, and C^+ joins L_0 to L_2. Thus the end-points 1, 0 of C^- and the end-points 0, 2 of C^+ are points of L_1, L_0, L_2, respectively. Let $F^-(x, y, y')$, $F^+(x, y, y')$ be any two functions defined, respectively, for all $(x, y) \in D_1^-$, $-\infty<y'<+\infty$, and $(x, y) \in D_1^+$, $-\infty<y'<+\infty$, and of class C^3. The necessary conditions for a minimum of the integral $J=\int_{C^-} F^- + \int_{C^+} F^+$ in the class $\{C\}$ have been investigated by the author in two previous papers [Dokl. Akad. Nauk SSSR **79** (1951), 565–568, 719–722; MR **13**, 474], in the line of the Weierstrass-Bliss theory with emphasis on the Jacobi condition, second variation $I_2(\eta, \tau)$, and the focal points 1′, 2′ of L_1, L_2, respectively, on an extremal E_{102}. In the present paper the author shows that the Bliss condition concerning the ordering of the points 102 and 1′, 2′ on E_{102}, or on a prolongation of E_{102} in D_0 and D_2, is sufficient in order that the second derivative $I_2(\eta, \tau)$ be positive.

L. Cesari (Baltimore, Md.)

4203:

Dedecker, P. Quelques applications de la suite spectrale aux intégrales multiples du calcul des variations et aux invariants intégraux. Bull. Soc. Roy. Sci. Liège **24** (1955), 276–295.

{Editor's note: In connection with reviews #4203, 4204, cf. the author's memoir "Calcul des variations et topologie algébrique," Mém. Soc. Roy. Sci. Liège (4) **19** (1957), no. 1 [MR **20** #2647].}

L'A. considère un groupe, anneau ou A-module muni d'un opérateur différentiel et filtré par les éléments d'un ensemble ordonné M (qui n'est pas nécessairement l'ensemble des entiers comme dans la théorie classique) et généralise la construction de la suite spectrale; il définit les termes $E(p, q, r, s)$ [Convegno Internaz. Geometria Differenz., Italia, 1953, pp. 247–262, Ed. Cremonese, Roma, 1954; MR **16**, 521] généralisant les quotients de la suite spectrale classique. L'ensemble des E et des homomorphismes Δ (différentielles homogènes) qui les relient constituent un diagramme à 4 dimensions que l'A. appelle diagramme spectral associé au groupe différentiel M-filtré. En considérant les homomorphismes dans un anneau A, l'A. introduit aussi une filtration duale et obtient deux diagrammes spectraux duaux. — L'A. traite d'une manière détaillée pour les chaines singulières cubiques [dues à J. P. Serre, Ann. of Math. (2) **54** (1951), 425–505; MR **13**, 574] la filtration résultant d'une structure fibrée; il montre l'aspect géométrique et intuitif de cette filtration.

G. Hirsch (Zbl **67**, 332)

4204:

Dedecker, P. Quelques applications de la suite spectrale aux intégrales multiples du calcul des variations et aux invariants intégraux. II. Bull. Soc. Roy. Sci. Liège **25** (1956), 387–399.

The author gives a resumé of some of the fundamental notions upon which he has based his very general approach to the calculus of variations and the theory of invariant integrals. Let x be a point of a topological space X, and U_x and V_x two open sets (containing x). If A_U is the ring of functions defined in an open set U, the identification of an element $f \in A_U$, and an element $g \in A_V$, which coincide in a set $W_x \subset U_x \cap V_x$ leads to the notion of local jet, the common jet $f_x=g_x$ of f and g in x. The set of these jets forms a ring A_x attached to x. The union A of all A_x, together with a topology defined on it, leads to the notion of stack. The set A is a particular case of a more general notion involving that of category due to Eilenberg and MacLane [Trans. Amer. Math. Soc. **58** (1945), 231–294; MR **7**, 109]. From this the author proceeds to define the local type Λ of mappings, the local class of type Λ and finally jet space. Stacks are then introduced as jet spaces with a certain particular structure which the author defines. Further notions are introduced with a view to later use, such as coherent substack of an analytic stack and finally the notion of costack.

E. T. Davies (Southampton)

GEOMETRIES, EUCLIDEAN AND OTHER

See also 3853, 3880, 4235, 4236, 4362, 4474.

4205:

Dubikajtis, L. On the incidence axioms of various geometries. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 423–427.

An attempt is made to coordinate the axiomatic

bases of various geometries as far as incidence relations are concerned [cf. Blaschke, Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie, vol. III, Springer, Berlin, 1929], and then to extend the basis in certain cases to include lattice properties. *G. de B. Robinson* (Toronto, Ont.)

4206:

Maravall Casesnoves, Dario. Generalization of the theorems of Guldin. Points for which the ellipsoid of inertia is a sphere. Gac. Mat., Madrid **9** (1957), 225–228. (Spanish)

4207:

Frame, J. S. Approximating a circular segment by use of Diophantine equations. Amer. Math. Monthly **65** (1958), 268–271.

A rational approximation is obtained for the ratio $G=S/T$ of the area S of a segment of a circle to the area T of the inscribed isosceles triangle, as a function of the ratio $h=H/R$ of the height H of the segment to the radius R of the circle. According to the author, this approximation is uniformly more accurate for $0<h\leqq 1$ than the approximation 22/7 for π, while it involves few terms or factors and only relatively small integral coefficients.

E. Frank (Chicago, Ill.)

4208:

Renzo, Gregorio. Del teorema di Steiner sul triangolo isoscele. Period. Mat. (4) **36** (1958), 110–114.

The author offers a direct elementary proof of the much discussed proposition: A triangle is isosceles if the internal bisectors of two of its angles are of equal length [Cf. Archibald Henderson, Scripta Math. **21** (1955), 223–232, 309–312]. *N. A. Court* (Norman, Okla.)

4209:

Hazanov, M. B. Some geometric approximations. Kabardin. Gos. Ped. Inst. Uč. Zap. **12** (1957), 8–13. (Russian)

Iterative constructions with ruler and compass with any desired accuracy are given for the following problems: construction of a radian and of a line segment of length π, quadrature of the circle, trisection of an angle, duplication of a cube, construction of the regular heptagon.

4210:

Manara, C. F. Sul concetto di equivalenza per i poligoni ed i poliedri. Period. Mat. (4) **35** (1957), 279–285.

4211:

Veldkamp, G. R. Still another generalisation of Pompeïu's theorem. Nieuw Tijdschr. Wisk. **45** (1957/58), 197–225. (Dutch)

Given an equilateral triangle ABC and a point P in its plane, then it is always possible to construct a triangle with its sides equal to PA, PB, PC. This theorem of Pompeïu is generalized in the following way. Given a plane polygon of $2n$ sides (or a plane convex polygon of any number of equal sides), then there always exists a polygon having as its sides the distances from an arbitrary point P in the plane to the midpoints of the sides of the given polygon. Reference is made to W. Boomstra, Nieuw Tijdschr. Wisk. **44** (1956/57), 285–288 [MR **20** #5449].

S. R. Struik (Cambridge, Mass.)

4212:

Hecquet, J., et Thébault, V. Sphères adjoints d'un tétraèdre. Mathesis **67** (1958), 125–131.

Given a tetrahedron $T=ABCD$ the authors associate four spheres with each pair of opposite edges of T, as follows:

(1) edges BC, DA; spheres $(AB)_1$ $(DC)_1$; $(CA)_1$, (DB_1),

(2) CA, DB; $(AB)_2$ $(DC)_2$; $(BC)_1$, $(DA)_1$,

(3) AB, DC; $(BC)_2$, $(DA)_2$; $(CA)_2$ $(DB)_2$,

where $(AB)_1$ is the sphere passing through the points A, B and tangent to the edges BC, DA; similarly for the other spheres.

The following are some of the results obtained. The spheres of each tetrad (1), (2), (3) form a coaxial net having for axis the bimedian of T relative to the respective pair of opposite edges of T; the three basic planes of the nets pass through the Monge point of T.

Each of the twelve spheres considered passes through two of eight Brocard points of the faces of T. Their twelve centers determine three parallelograms and the three lines joining the centers of those parallelograms to the circumcenter of T are parallel to the bialtitudes of T.

The centroid G of T is the center of three spheres respectively orthogonal to the spheres of the three nets. If T is orthocentric, those three spheres coincide with the sphere having G for center and orthogonal to the polar sphere of T. *N. A. Court* (Norman, Okla.)

4213:

Myller, A. Un chapitre d'une géométrie de la récurrence. Suites de triangles dérivés de deux triangles donnés. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **2** (1956), 1–14. (Romanian. Russian and French summaries)

L'auteur considère une suite de triangles T_k tous circonscrits à un triangle donné et tels que T_k est circonscrit à T_{k-1}. *O. Bottema* (Delft)

4214:

Kooistra, R. Trigonometric inequalities in a triangle. Nieuw Tijdschr. Wisk. **45** (1957/58), 108–115. (Dutch)

The author derives upper and lower bounds for sums and products over the three angles of a triangle of the functions $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\cot\alpha$, $\sin^2\alpha$, $\cos^2\alpha$, $\tan^2\alpha$, $\cot^2\alpha$, $\sec^2\alpha$, $\operatorname{cosec}^2\alpha$ and for the same functions of $\alpha/2$. Some of these are derived by a transformation of the Weitzenböck-inequality (1919): $a^2+b^2+c^2\geqq 4O\sqrt{3}$ {which can easily be verified from $(a^2+b^2+c^2)^2-48O=a^4+b^4+c^4-a^2b^2-b^2c^2-a^2c^2\geqq 0$}. The bounds are sharpened for acute-, obtuse- and right-angled triangles using the corresponding inequalities $s\gtreqless 2R+r$ {which condition could have been simply derived by multiplying the obvious condition $\cos\gamma/2-\sin\gamma/2\gtreqless 0$ for $\gamma>\alpha>\beta$ into the always positive quantity $\cos\frac{1}{2}(\alpha-\beta)-\cos\frac{1}{2}\gamma$, leading to $\sin\alpha+\sin\beta+\sin\gamma\gtreqless 2+(\sin\alpha+\sin\beta-\sin\gamma)\tan\frac{1}{2}\gamma$, and so to $s\gtreqless 2R+r$}. *E. M. Bruins* (Amsterdam)

4215:

Toscano, Letterio. Circonferenze notevoli di Schoute. Giorn. Mat. Battaglini (5) **5** (**85**) (1957), 209–243.

The author computes and lists the algebraic values of the measures of many angles, lengths, and tripolar distances of points of interest in the study of triangles and their associated circles.

E. F. Beckenbach (Los Angeles, Calif.)

4216:

Cavallaro, Vincenzo G. Triangoli $T'=x\cdot T$. Archimede **10** (1958), 115–118.

The three equilateral triangles constructed outside [resp. inside] on the sides of a triangle ABC bring its "Torricelli segments" T [resp. T'] into evidence [Cavallaro, Giorn. Mat. Battaglini (5) **4(84)** (1956), 81–91; MR **18**, 592] and $T'=x(T)$ where $x<1$. The relationships between centroid, Lemoine point, Brocard points, etc. take on simple forms if, e.g., $x=\frac{1}{2}(\sqrt{5}-1)$ or $x=\frac{1}{2}$.

S. R. Struik (Cambridge, Mass.)

4217:

Hebroni, Pessach. On the equilateral triangle as an extremal triangle. Riveon Lematematika **11** (1957), 61–69. (Hebrew)

The author applies (essentially) the principle that a symmetric function $f(x_1, \cdots, x_n)$ which has a unique extremum must assume it for $x_1=\cdots=x_n$, to prove some of the extremality properties of the equilateral triangle.

E. G. Straus (Los Angeles, Calif.)

4218:

Mallison, H. V. Pedal circles and the quadrangle. Math. Gaz. **42** (1958), 17–20.

Given four coplanar points A, B, C, D, let (A), (B), (C), (D), denote the pedal circles of those points, respectively, for the triangles formed by the remaining three points.

The author proves: The centers of the four pedal circles form a quadrangle inversely similar to the quadrangle $ABCD$; the common chord of the circles (A), (D) bisects the segments BH_1, CH_2 joining the points B, C to the orthocenters H_1, H_2 of the triangles BAD, CAD; and other properties.

Both analytic and synthetic proofs are used.

N. A. Court (Norman, Okla.)

4219:

Havel, Václav. Fundamentalsätze der mehrdimensionalen Zentralaxonometrie. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **7** (1957), 94–107. (Czech. Russian and German summaries)

4220:

Broch, E. K. A mathematical study concerning the three-dimensional lattice and close-packing of equal spheres. Avh. Norske Vid. Akad. Oslo. I. **1957**, no. 3, 7 pp.

The author shows that if the cubic close-packing of equal spheres is dislocated by a reflection without disturbing its density, the reflecting plane must be orthogonal to a trigonal axis of symmetry. In other words, this is the kind of dislocation that provides the transition to hexagonal close-packing [cf. Barlow, Nature **29** (1883), 186–188; W. W. R. Ball, Mathematical recreations and essays, Macmillan, London, 1939, p. 150; for a review of 1947 American edition, see MR **8**, 440; S. Melmore, Nature **159** (1947), 817; MR **9**, 53; A. P. Dempster. Canad. J. Math. **9** (1957), 232–234; MR **19**, 16; W. G. Burgers, Nederl. Tijdschr. Natuurk. **22** (1956), 245–270; MR **18**, 348].

H. S. M. Coxeter (Toronto, Ont.)

4221:

★Keller, Ott-Heinrich. Analytische Geometrie und lineare Algebra. Hochschulbücher für Mathematik. Bd. 26. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. xi+442 pp. (Insert: stereoscope with 77 diagrams) DM 42.40.

"The book is intended to introduce beginners to geometric and algebraic thought, so that it proceeds constantly from the concrete to the abstract. Axiomatics are omitted. An attempt is made to clarify the algebraic kernel of geometric concepts and methods of proof." (From the introduction.) There are 16 chapters with the headings: Koordinaten, Vektorrechnung, Geometrie der Einzelfiguren, lineare Algebra, Räume der Dimension n, Transformationen, orthogonale Transformationen, Affinitäten, Büschel und Bündel, Dualität und Liniengeometrie, Schnittpunktsätze, Doppelverhältnisse, Projektivität in Punktreihe und Strahlenbüschel, Kreisbüschel, projektive Transformationen (projektive Koordinaten), projektive Geometrie der Quadriken.

4222:

Szász, Pál. Independent elementary construction of the analytic geometry of the hyperbolical plane on the basis of Hilbert's "calculus of ends". Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 423–438. (Hungarian)

The author gives an elementary construction of the analytic geometry of the hyperbolic plane based on the "calculus of ends" of Hilbert. His method makes no use of Euclidean geometry and of hyperbolic trigonometry, but contains this latter as a consequence. For a more detailed exposition of the same construction see the paper reviewed below [#4223].

A. Kertész (Debrecen)

4223:

Szász, Paul. Unmittelbare Einführung Weierstrass-scher homogenen Koordinaten in der hyperbolischen Ebene auf Grund der Hilbertschen Endenrechnung. Acta Math. Acad. Sci. Hungar. **9** (1958), 1–28.

Unter einer hyperbolischen Ebene ist in dieser Abhandlung irgendeine Gesamtheit von 'Punkten' und 'Geraden' verstanden, für die ausser den ebenen Axiomen I, II, III von D. Hilbert [Grundlagen der Geometrie, 7. Aufl., Teubner, Leipzig-Berlin, 1930; Anhang III, 160–162] noch die folgenden zwei Axiome des Verfassers gelten: IV_1. "Es seien P, Q zwei verschiedene Punkte in der Ebene und QY eine Halbgerade an der einen Seite der Geraden PQ. So gibt es stets eine Halbgerade PX an derselben Seite von PQ, die QY nicht schneidet, während jede im $\angle QPX$ gelegene innere Halbgerade PZ diese Halbgerade QY schneidet." IV_2. "Es gibt eine Gerade g_0 und einen nicht auf ihr liegenden Punkt P_0 der Ebene derart, dass durch P_0 zwei verschiedene Geraden gelegt werden können, die g_0 nicht schneiden." (Also ist die hiesige hyperbolische Geometrie von derjenigen von W. Klingenberg [Math. Ann. **127** (1954), 340–356; MR **15**, 893] verschieden.) Dieses Axiomsystem ist äquivalent Szász, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 163–184; MR **20**#240; auch die folgende Referierung] mit demjenigen von D. Hilbert [Math. Ann. **57** (1903), 137–150], welches aus I, II, III und dem Axiom, "Ist g eine beliebige Gerade und P ein nicht auf ihr gelegter Punkt, so bilden die durch P gelegten und g schneidenden Geraden, die inneren Geraden eines gewissen $\angle(p_1, p_2)$," besteht. (Dabei sind p_1 und p_2 gewisse durch P hindurchgehende Geraden.) Dabei sind die Stetigkeitsaxiome nicht herangezogen. Während der von D. Hilbert gestreifte Weg zum Aufbau der hyperbolischen Geometrie der Ebene mit Hilfe seiner 'Endenrechnung' durch die projektive Geometrie führt, wird in dieser Abhandlung durch die unmittelbare Einführung gewisser homogenen Koordinaten und selbständige Begründung der analytischen Geometrie der hyperbolischen Ebene die Grundlage für einen vollständig elementaren Aufbau geschaffen. Diese Koordinaten werden bei der

Annahme der Stetigkeitsaxiomen (statt des Axioms IV_1), die das obige unvollständige Axiomsystem zu einem vollständigen machte, identisch mit den bekannten Weierstrassschen Koordinaten (bei der Annahme des Stetigkeitsaxioms hat der Verfasser eine unmittelbare Einführung der Weierstrassschen Koordinaten dargetan [siehe Acta Math. Acad. Sci. Hungar. **8** (1957), 139–157; MR **19**, 445]), deshalb hat der Verfasser sie ebenso genannt. Bei diesem Aufbau hat er mit der hyperbolischen Trigonometrie nichts zu tun, letztere eine Folge der hiesigen analytischen Geometrie seiend. [Vgl. M. Dehn, Math. Ann. **53** (1900), 404–439, wo die hyperbolische Geometrie unter den Axiomen I, II, III Hilberts ohne Heranziehung der Stetigkeitsaxiome durch Betrachtung der Winkelsumme des Dreiecks und der Anzahl der durch einen Punkt zu einer Geraden gelegten Parallelen untersucht ist.] *T. Takasu* (Yokohama)

4224:

Szász, Paul. A remark on Hilbert's foundation of the hyperbolic plane geometry. Acta Math. Acad. Sci. Hungar. **9** (1958), 29–31.

Der Satz Hilberts [siehe obige Referierung letzter Abhandlung desselben Verfassers], welcher dem Parallelitätsaxiom entspricht, ist mittels der Axiome I, II, III D. Hilberts und der IV_1, IV_2 des Verfassers bewiesen. Dazu hat der Verfasser den folgenden Hilfssatz Janos Bolyais [Appendix, Scientiam spatii absolute veram exhibens ..., Maros-vásárhely, 1832; änglische Übersetzung in "Non-Euclidean Geometry", Dover, New York, 1955; § 1] aus seinem Axiom IV_1 (vgl. die letzte Abhandlung) hergeleitet: "Es seien P und Q irgend zwei Punkte einer Ebene. Es sei QY irgendeine Halbgerade. Bewegt ein Punkt R auf der Halbgeraden QY, so ist $\angle QRP \to 0$ für $\overline{QR} \to +\infty$." *T. Takasu* (Yokohama)

4225:

Černyaev, M. P. Elements of a synthetic theory of oblique ruled surfaces of second order. Rostov. Gos. Ped. Inst. Uč. Zap. **4** (1957), 43–52. (Russian)

4226:

Moscovici, M. Une interprétation géométrique naturelle des solutions complexes de quelques problèmes de géométrie analytique. Com. Acad. R. P. Romîne **7** (1957), 407–412. (Romanian. Russian and French summaries)

A continuation of the article [same Com. **5** (1955), 943–947] listed in MR **17**, 398.

4227:

Cardoso, Jayme Machado. A note on affinity in rotation. Soc. Parana. Mat. Anuário **4** (1957), 47–48. (Portuguese)

If a plane α is rotated through an angle θ about a vertical axis r, there exists an affinity between the vertical Mongean projections of the elements of α before and after rotation. *R. G. Stanton* (Waterloo, Ont.)

4228:

Hofmann, L. Über eine elementargeometrische Aufgabe, die auf ein klassisches Problem der Geometrie führt. Elem. Math. **13** (1958), 49–55, 79–85.

The paper deals with the following question of elementary geometry: In each of two planes ε and ε', which may also coincide, let five points P_k and P_k', respectively, be given ($k=1, \cdots, 5$); to find two points S and S' in ε and ε', respectively, such that the ordered pencil of the lines SP_k be congruent to the ordered pencil of the lines $S'P_k'$, each P_k corresponding to P_k'. This is equivalent to the requirement that the ordered pencil ($SP_1, \cdots, SP_5, SJ, SK$) be projectively related either to ($S'P_1', \cdots, S'P_5', S'J', S'K'$) or to ($S'P_1', \cdots, S'P_5', S'K', S'J'$), where J, K and J', K' are the absolute points of ε and ε', respectively. Thus the question reduces to the problem of finding two centers S and S' which form two projectively related pencils ($SQ_1, \cdots, SQ_7$) and ($S'Q_1', \cdots, S'Q_7'$) with given points $Q_1, \cdots, Q_7, Q_1', \cdots, Q_7'$. This is a classical problem [M. Chasles, Nouv. Ann. Math. **14** (1855), 211]. The author gives a general solution of this problem and applies it to working out a numerical example of the above-mentioned question. *R. Artzy* (Haifa)

4229:

Deaux, R. Coniques analogues aux cercles de Tücker. Mathesis **67** (1958), 113–124.

If a given triangle T is transformed into a triangle T' by the homology (S, d, k), where S, d, k are the center, axis, and constant of homology, the sides of T' meet the non-homologous sides of T in six points of a conic Γ_k.

The author studies analytically the family of conics Γ generated by Γ_k when k varies while S and d remain fixed. He establishes the following properties, among others.

Let s denote the harmonic polar of S for T. The point $P=sd$ has the same polar for all the conics of the family Γ. A line through P meets the conics Γ in pairs of points of an involution of which P is a double point.

Let D be the harmonic pole of d for T and (D) the conic inscribed in T at the feet of the cevians of D. If the points S, D lie in different regions into which the plane is divided by the conic (D), there exist two homologies (S, d) the product of whose constants is equal to 1 such that each of the corresponding conics T' is triply perspective to T by a circular permutation of its vertices.

If d coincides with the line at infinity, and S (a) coincides with the Lemoine point of t; (b) lies on the inscribed Steiner ellipse of T; (c) lies on the orthic axis of T; then the conics Γ are (a) circles (Tücker); (b) parabolas; (c) equilateral hyperbolas. *N. A. Court* (Norman, Okla.)

4230:

Mammana, Carmelo. Determinazione dei tipi di omografie di cui una data omografia si può considerare come limite. Ann. Scuola Norm. Sup. Pisa (3) **11** (1957), 249–263.

E'noto che un'omografia particolare dello spazio S_n si può considerare in varî modi come limite di omografie generali di S_n. L'autore considera un'omografia ω di S_n, di assegnata caratteristica di Predella, e determina tutte le omografie $\omega(t)$ di S_n (con coefficienti funzioni di un parametro t) per le quali avvenga che $\lim_{t\to 0} \omega(t)=\omega$ (intendendo che il fattore di proporzionalità a meno del quale son determinati i coefficienti di $\omega(t)$ sia tale che per $t\to 0$ i coefficienti di $\omega(t)$ tendano ai corrispondenti coefficienti di ω), assegnando delle condizioni necessarie e sufficienti a cui devono soddisfare le caratteristiche di Predella delle $\omega(t)$. *D. Gallarati* (Genova)

4231:

van Est, W. T. A group theoretic interpretation of area in the elementary geometries. Simon Stevin **32** (1958), 29–38.

Let G denote the group of displacements (i.e., direct isometries) in a Euclidean or non-Euclidean plane. Let each element g_i of G transform a certain fixed point into

p_i. Let $F(g_1, g_2, g_3)$ denote the oriented area of the triangle $p_1p_2p_3$. The author points out that, since $F(g_0g_1, g_0g_2, g_0g_3)=F(g_1, g_2, g_3)$ and $F(g_1, g_2, g_3)-F(g_0, g_2, g_3)+F(g_0, g_1, g_3)-F(g_0, g_1, g_2)=0$, the function F may be regarded as a homogeneous cocycle on G, determining a central group extension of G by the additive group C of the real numbers. To prove that this extension is not merely a direct product $G\times C$, he finds a commutator in this extension which is a non-zero element of C. For this purpose he uses, in the hyperbolic case, the translations a, b, c, d, which generate the fundamental group

$$abcda^{-1}b^{-1}c^{-1}d^{-1}=1$$

of the surface of genus 2, in the manner described by Coxeter and Moser [Generators and relations for discrete groups, Springer, Berlin-Göttingen-Heidelberg, 1957; MR 19, 527; p. 60]. A deeper investigation of the hyperbolic case makes use of the fact that the product of translations along the three sides of a triangle is a rotation through an angle equal to the area of the triangle.

H. S. M. Coxeter (Toronto, Ont.)

4232:

Bilinski, Stanko. Über eine gewisse Kurvenzuordnung in der hyperbolischen Ebene. Comment. Math. Helv. 32 (1957), 1–12.

In hyperbolic plane geometry, the curvatures of a true circle, a horocycle, and an equidistant curve, are, respectively, >1, $=1$, <1; an arc of a curve E in which the curvature k is <1 has no true centres of curvature or evolute, E being osculated at each point of the arc by an equidistant curve. To replace these the author considers, for each point of such an arc, the "basis point" or intersection of the normal with the axis of the corresponding equidistant curve, and the "basoid" or locus of the basis point. At a point of E where $k=1$, separating arcs in which $k>1$, $k<1$, the normal is a common asymptote to the evolute of the former arc and the basoid of the latter; at an inflexion of E, the basoid intersects E.

The chief metrical result is that the area between two normals to E and the arcs of E and the basoid cut off between these is given by $\int_{s_1}^{s_2} k\,ds$, s being arc length on E, and s_1, s_2 its values at the beginning and the end of the arc in question.

P. Du Val (London)

CONVEX SETS AND DISTANCE GEOMETRIES

4233:

Beale, Martin; et Drazin, Michael. Sur une note de Farquharson. C. R. Acad. Sci. Paris 243 (1956), 123–125.

One method for handling a non-zero-sum n-person game is to imbed it in a zero-sum game with an additional "fictitious" player. The payoff to this player is defined to be the negative of the sums of the payoffs to the other players. It is shown that an analogous thing can be done for games with any number of players, finite or infinite, in which the payoff functions take values in an arbitrary ordered set. Let Q be the set of players and, for a in Q, let S_a be a's strategy set. Then each player orders the set $X=\prod_{a\in Q} S_a$ by a transitive order relation. The game is called a "jeu d'ophélimité" when, for any elements x, y in X, if x is preferred to y by some player, then y is preferred to x by some other (one man's meat is another man's poison).Clearly, the orderings induced by numerical payoff functions have this property in the zero-sum case. It is shown that any game can be imbedded in a jeu d'ophélimité with an additional fictitious player having a single strategy. The proof is an easy exercise for the case of a finite number of players and a straightforward extension via well-ordering in the infinite case.

D. Gale (Providence, R.I.)

4234:

Goldman, A. J. The probability of a saddlepoint. Amer. Math. Monthly 64 (1957), 729–730.

It is shown that if the elements of the $m\times n$ payoff matrix are independent random variables with the same continuous distribution function, then the probability that the game has a saddle-point is $m!n!/(m+n-1)!$

M. Dresher (Pacific Palisades, Calif.)

4235:

Eggleston, H. G. Figures inscribed in convex sets. Amer. Math. Monthly 65 (1958), 76–80.

The author notes that the following results are known concerning convex bodies (closed bounded convex sets with interior points). 1. For each planar convex body there is an inscribed square. 2. For each point interior to a solid convex body there is an inscribed square with that point as center. 3. A regular octahedron may be inscribed in every solid central convex body. Examples are then given as follows which limit generalizations of these results. A. A planar convex body of constant width in which no regular n-gon may be inscribed for $n>4$. (The Reauleaux triangle is shown to have this property.) B. A solid convex body such that for $n>4$ there exists a point p so that no regular n-gon with center p is inscribed in the body. (A solid cylinder based on the Reuleaux triangle is the example given.) C. A solid central convex body in which a cube cannot be inscribed. (This body is selected as the intersection of an ellipsoid with a certain slab.)

P. C. Hammer (Madison, Wis.)

4236:

Lenz, Hanfried. Einige Anwendungen der projektiven Geometrie auf Fragen der Flächentheorie. Math. Nachr. 18 (1958), 346–359.

This paper indicates applications of the theory of real projective geometry in proving theorems concerning convex bodies and in establishing results concerning affine surface theory. In particular, several characterizations of ellipsoids and of quadric surfaces are given. The proofs, with the aid of several lemmas from projective geometry, are generally simple. In some cases, as the author notes, differentiability conditions are imposed which are not necessary.

P. C. Hammer (Madison, Wis.)

4237:

Melzak, Z. A. Limit sections and universal points of convex surfaces. Proc. Amer. Math. Soc. 9 (1958), 729–734.

Let S denote a closed strictly convex surface in E^3. For a point $x\in S$ consider sequences of planes P_n intersecting S and tending to a supporting plane of S at x. If $P_n\cap S=C_n$, let λ_nC_n be similar to C_n in the ratio λ_n:1. The set C^x of convex curves consists of all curves which are limits of sequences of curves λ_nC_n. If the P_n are restricted to planes parallel to a fixed supporting plane of S at x we denote the corresponding set of limit curves by C_p^x. The set C^x or C_p^x is called universal if it contains a curve congruent to any given closed convex curve in E^2. The following facts are proved. There exists an S of class C^∞ except at q where it is differentiable and C_p^q is universal. There is an S such that C^x is universal for all $x\in S$. There

is an S and a $q \in S$ such that $C_p{}^q$ is empty but C^q is universal. There is an S where $C_p{}^x = \{\lambda K(x)\}$ with $K(x)$ a fixed closed convex curve for each $x \in S$, but C^q is universal for a suitable $q \in S$.

H. Busemann (Cambridge, Mass.)

4238:

Stewart, B. M. Asymmetry of a plane convex set with respect to its centroid. Pacific J. Math. **8** (1958), 335–337.

If K is a bounded convex set, and $P \in K$, let $S(P)$ be the intersection of K with the radial reflection of K through P. $S(P)$ is a central convex set with P as center. Let $f(P) = \text{meas}(S(P))/\text{meas}\, K$. It is known that the level surfaces (curves) of f are convex [Fary and Redei, Math. Ann. **122** (1950), 205–220; MR **12**, 526; S. Stein, Pacific J. Math. **6** (1956), 145–148; MR **18**, 228]. In the case of planar sets K, Besicovitch showed that there exist points $P \in K$ with $f(P) \geqq 2/3$. [J. London Math. Soc. **23** (1948), 237–240; MR **10**, 320]. The reviewer and Ellen F. Buck [Math. Mag. **22** (1949), 195–198; MR **10**, 621] showed that any set K contains a six-partite point P_0 through which three lines can be passed dividing K into six parts of equal area, and Eggleston [J. London Math. Soc. **28** (1953), 32–36; MR **14**, 896] showed that for any six-partite point P_0, $f(P_0) \geqq 2/3$. In the present paper, the author shows that $f(G) \geqq 2/3$, where G is the centroid of K. It is known that unless K is a triangle, max $f(P) > 2/3$; the author observes that the (unique) point P^* which maximizes f will not, in general, be G. [See also Lyusternik, Convex figures and polyhedra, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; MR **19**, 57.]

R. C. Buck (Stanford, Calif.)

4239:

Rubin, Herman; and Wesler, Oscar. A note on convexity in Euclidean n-space. Proc. Amer. Math. Soc. **9** (1958), 522–523.

It is a fundamental fact about convex sets in Euclidean n-space that a finite linear combination of points of the set with non-negative weights summing to one is in the set. The authors prove that this remains so when infinite linear combinations are taken, with non-negative weights forming a convergent series of sum one, or even when the sum is replaced by a Stieltjes integral with respect to a probability measure. The result does not remain true for spaces not of finite dimension.

J. W. Green (Los Angeles, Calif.)

4240:

Dinghas, Alexander. Zur Einzigkeitsfrage der Minkowski-Lusternikschen Ungleichung für die Relativoberfläche. Math. Z. **68** (1957), 299–315.

The classical Brunn-Minkowski theorem states that, if A, B are convex subsets of E^n, and μ_* is the inner Lebesgue measure, then

$$(1) \qquad \mu_*(A+B) \geqq [\mu_*(A)^{1/n} + \mu_*(B)^{1/n}]^n.$$

The equality is strict unless A, B are homothetic. This result was generalized to arbitrary sets by Lusternik, and in the general case, equality holds only if A, B are convex homothetic sets with perhaps a set of measure zero removed, provided that $\mu_*(A)$ and $\mu_*(B)$ are positive.

If the B-surface-area of A is defined by the relation $M(A|B) = \liminf\, [\mu_*(A+hB) - \mu_*(A)]/h$, then it follows from (1) that (2) $M(A|B) \geqq (\mu_*(A))^{1-1/n}(\mu_*(B))^{1/n}$. If B is a sphere and A is convex, the quantity $M(A|B)$ is the classical surface area of the frontier of A, and (2) is the classical isoperimetric inequality.

Since (2) is derived from (1) by a limiting process, it does not follow at once that strict inequality holds in (2) whenever it holds in (1). In the paper under review, this is shown to be true provided that A, B have the properties (i) $\mu_*(A) > 0$, $\mu_*(B) > 0$, and (ii) if $\bar{A} - (\bar{A}_*)$ isn ot empty then its projection on some hyperplane has positive measure (here A_* is the set of points of inner density of A).

The method used is an ingenious combination of results due to Henstock and the reviewer [Proc. London Math. Soc. (3) **3** (1953), 182–194; MR **15**, 109] and methods due to the author and Erhardt Schmidt [Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1943**, no. 7; MR **8**, 338] in their work on the case when B is a sphere.

The paper also includes a simple direct proof of (1), based on Minkowski's ideas. *A. M. Macbeath* (Dundee)

4241:

Radó, Tibor. On the space of oriented lines in Euclidean three-space. Amer. Math. Monthly **64** (1957), no. 8, part II, 79–89.

Let G denote the set of oriented lines g of euclidean three-space E_3. If $g \in G$, it may be represented by any ordered pair x, y of its points, where the direction from x to y agrees with the orientation of g. A limit concept is introduced in G by defining g_0 to be a limit of a sequence $g_1, g_2, \cdots$ of elements of G if and only if representations $\gamma(x_n, y_n)$ of g_n exist ($n = 0, 1, \cdots$) such that each of the two sequences of euclidean distances x_0x_n, y_0y_n has limit zero. The resulting space is metrized upon defining the distance $d(g_1, g_2)$ of two arbitrary elements g_1, g_2 of G by $d(g_1, g_2) = |u(g_2) - u(g_1)| + |\varphi(g_2) - \varphi(g_1)|$, where $u(g)$ is the unit vector from the origin agreeing with g in direction and sense, and $\varphi(g)$ is the vector from the origin to the nearest point of g. Though the limit concept defined in G is invariant under all congruences ρ of E_3 with itself, the metrizing distance function is not invariant, i.e., it is not the case that $d(g_1, g_2) = d[\rho(g_1), \rho(g_2)]$, where $\rho(g)$ is the oriented line into which g is carried by ρ (if $g = \gamma(x, y)$ then $\rho(g) = \gamma[\rho(x), \rho(y)]$). The author shows that the only invariant distance functions in G are (1) isolated: $\delta_i(g_1, g_2)$ equals 0 if $g_1 = g_2$, and equals 1 otherwise; and (2) semi-isolated: $\delta_{si} = d_e(g_1, g_2)/[1 + d_e(g_1, g_2)]$ if g_1, g_2 are parallel with the same sense, and $\delta_{si} = 1$, otherwise, where $d_e(g_1, g_2)$ denotes the minimum (euclidean) distance of the two (non-directed) lines that carry g_1, g_2. The corresponding two topologies are not fruitful. {Reviewer's note. What the author terms a scale function was called by the reviewer a metric transform [see, for example, Ergeb. Math. Kolloq. **7** (1936), 8–10]. Though, as the author proves, metric transforms of E_2 either leave the topology of E_2 invariant or change it to the trivial (discrete) one, numerous metric transforms of E_2 are metrically quite interesting}.

L. M. Blumenthal (Columbia, Mo.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 3980, 4171, 4261, 4265, 4269, 4270, 4271.

4242:

Besicovitch, A. S. On homoeomorphism of perfect plane sets. Proc. Cambridge Philos. Soc. **54** (1958), 168–186.

The author considers plane bounded perfect sets and gives characteristic properties of two homeomorphic classes of such sets. One starts by giving a new proof of the following theorem by Antoine: If F, ϕ are two plane

bounded perfect sets such that each component of F and of ϕ is a point, then there exists an autohomeomorphism of the plane mapping F onto ϕ [L. J. Antoine, J. Math. Pures Appl. (8) 4 (1921), 221–326]. This theorem is then generalized by considering perfect sets all of whose components are simple arcs. Let AB, $A'B'$ be 2 simple arcs with their end-points; the Fréchet distance $\Delta_F(AB, A'B')$ is defined as $\inf_h \sup_{M\in AB} MhM$, h running over the class of all the homeomorphisms between AB and $A'B'$; MhM means the length of the segment from M to hM. A plane perfect arc-set is defined as any plane set whose components are simple arcs such that the set of these arcs in Fréchet's metric is compact, closed and has no isolated elements. The preceding theorem is transferrable to perfect arc-sets (Theorem 2). The proof takes 15 pages (171–184) and fundamentally is based on a fine generalization to perfect arc-sets of the following lemma. For each number $\varepsilon>0$ and each plane bounded perfect set F whose components are points there exists a finite family of closed Jordan domains covering F, each being of a diameter $<\varepsilon$ and containing a perfect subset of F. According to theorem 3, any plane perfect arc-set S is characterized by the following 5 properties: (1) S is plane and perfect; (2) All components of S are Jordan arcs; (3) The closure of any subset of components of S is a subset of components of S; (4) For each component x of S, one has $\overline{S\setminus x}=S$; (5) There exists a function $f(l)>0$ of a positive variable such that $\lim_{l\to 0} f(l)=0$ and that $\operatorname{diam} \gamma<f(l)$ for each arc γ of S subtended by a chord of length $\leq l$. *Đ. Kurepa* (Zagreb)

4243:

Drăgilă, Pavel. Définition générale des transformations topologiques dans le plan et dans l'espace. Bull. Soc. Roy. Sci. Liège 27 (1958), 157–160.

Theorem: A transformation of the plane onto itself is topological if and only if: (1) It carries Jordan curves, and only Jordan curves, into Jordan curves: (2) it preserves like orientation on two curves with like orientation.

An analogous theorem holds in 3-space if Jordan curve is replaced by Jordan surface, where the latter is a closed oriented surface cut by any plane in a union of Jordan curves. *M. E. Shanks* (Lafayette, Ind.)

4244:

Tanaka, Tadashi. A set-theoretical characterization of the torus. J. Sci. Hiroshima Univ. Ser. A 21 (1957/58), 119–124.

The author gives a set-theoretic characterisation of the torus as a locally connected, compact, metric continuum $T=J_1+J_2+R$, where: (a) J_1, J_2 are Jordan curves intersecting in a single point p, and $R=T-(J_1+J_2)$ is connected; (b) J_i separates irreducibly some connected neighbourhood of J_i into two regions which are locally connected at any point of J_i; and (c) if X is any spanning arc of J_1+J_2 in T, then X separates R irreducibly, and if Y is a second such spanning arc which covers one end of X, then Y separates R between two points of X which are sufficiently near to a and b, respectively. The definition of "covering one end of X" is too long to write out in a short review. The methods depend on known characterisations of 2-cells and 2-spheres.

H. B. Griffiths (Bristol)

4245:

Koutský, Karel. Sur la détermination des espaces topologiques par des systèmes complets d'entourages des points. Publ. Fac. Sci. Univ. Masaryk 1956, 153–163. (Czech. Russian and French summaries)

If P is a given set, $M\subset P$, u a function defined on $\{M\}$, such that $u(M)\subset P$, then u defines a "topology (P, u)". If $x\in P$ does not belong to all $u(M)$, it is called an O-point and $\{P-S|x \notin u(S)\}$ is a system of all its neighborhoods. A complete system O_x of neighborhoods of x is defined as usual. Suppose P and a system O_x of subsets of P are given; then it can be shown that there exists (at least) one topology (P, u) such that: (i) $x\in P$, $O_x=\emptyset$ implies that x is not an O-point; (ii) $x\in P$, $O_x\neq\emptyset$ implies that x is an O-point and O_x is a complete system of neighborhoods of x in (P, u). In this paper the set of topologies is studied which satisfy these two conditions. *František Wolf* (Berkeley, Calif.)

4246:

Sekanina, Milan. Les systèmes complets d'entourages des ensembles dans les espaces topologiques généraux. Publ. Fac. Sci. Univ. Masaryk 1956, 185–192. (Czech. Russian and French summaries)

In a given topological space (P, u) let $Q_u(X)$ denote the system of neighborhoods of an $X\subset P$, given by the topology u. The author studies the structure of γ, the family of topologies v such that $Q_u(X)$ is a complete set of neighborhoods of X for the topology v. He introduces the concept of the topology of type sM. The topology w is of type sM if, for all M_1, M_2 such that $M_1\subseteq M_2\subset P$, the relation $(M_1)_{w^i} \supsetneqq (M_2)_{w^i}$ is incorrect, where M_{w^i} denotes the set of interior points of M under w. The author proves that the set of minimal elements in γ is the set of topologies of type sM. *František Wolf* (Berkeley, Calif.)

4247:

Mrówka, S. On the potency of compact spaces and the first axiom of countability. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 6 (1958), 7–9.

Theorem. Suppose that at no point of the compact space X is there a countable base for the topology; then the cardinality of X is at least $2^{\aleph_1}$.

M. E. Shanks (Lafayette, Ind.)

4248:

Papić, Pavle. Quelques propriétés des espaces totalement ordonnés et des espaces de la classe R. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke 6 (302) (1957), 171–196. (Serbo-Croatian. French summary)

This paper contains results about separation of sets, metrization of totally ordered spaces, and spaces E having ramified bases of neighborhoods. Typical results: Every such space E has a ramified base in which each member has at most two immediate successors. Suppose that E has a ramified base T such that the intersection of any chain of noncompact members of T, if a member of T, is noncompact; then E is homeomorphic to a totally ordered space. *L. Gillman* (Princeton, N.J.)

4249:

Onuchic, Nelson. On two properties of P-spaces. Portugal. Math. 16 (1957), 37–39.

A completely regular space E is called a P-space if every prime ideal in $C(E)$ is maximal; E is a P-space if and only if every zero-set is open, and if and only if every G_δ is open [Gillman and Henriksen, Trans. Amer. Math. Soc. 77 (1954), 340–362; MR 16, 156]. Theorem 1: E is a P-space if and only if every pointwise convergent sequence of func-

tions in $C(E)$ has a continuous limit. The proof is straightforward. Theorem 2: If (*) every separately continuous real-valued function on $E\times E$ is continuous, then E is discrete. The proof of theorem 2 is based upon two lemmas. Lemma 1: If (*), then E is a P-space. Lemma 2: If E is any Hausdorff space with a base of open-and-closed sets, and if (*), then E is discrete. *L. Gillman* (Princeton, N.J.)

4250:
Iséki, Kiyoshi. A characterisation of P-spaces. Proc. Japan Acad. **34** (1958), 418–419.

The author proves the easy theorem that a completely regular space is a P-space if and only if the limit of any convergent sequence of continuous functions is continuous. In a note added in proof, he remarks that the result is observed by N. Onuchic [see preceding review].
L. Gillman (Princeton, N.J.)

4251:
McDowell, Robert H. Extension of functions from dense subspaces. Duke Math. J. **25** (1958), 297–304.

The problem is the description of conditions for the continuous extension of f: $X\to Y$ over E, where X is dense in E. Descriptions are given in terms of Stone-Čech compactifications; there is no essential novelty. The following application seems new: a sufficient condition is that E be extremally disconnected and Y compact. The presentation is independent of the work of Stone and Čech. *J. Isbell* (Seattle, Wash.)

4252:
Smirnov, Yu. A completely regular non-semibicompact space with a zero-dimensional Čech complement. Dokl. Akad. Nauk SSSR **120** (1958), 1204–1206. (Russian)

For separable metric spaces, semibicompactness (rim-compactness) is known to be necessary and sufficient for the existence of a compactification in which the complement of the given space has ind zero. It is sufficient in any case (the Freudenthal compactification). The author shows that it is not necessary, starting with Dowker's example M [Quart. J. Math. Oxford Ser. (2) **6** (1955), 101–120; MR **19**, 157] for which ind $M=0<\dim M$.
J. Isbell (Seattle, Wash.)

4253:
Sklyarenko, E. Bicompact extensions of semibicompact spaces. Dokl. Akad. Nauk SSSR **120** (1958), 1200–1203. (Russian)

The principal result is a generalization of Freudenthal's theorem, that a space R has a compactification B with $\mathrm{ind}(B-R)=0$ if and only if R is semibicompact, from separable metric R to a class including the metric spaces, specifically spaces in which every compact subset has a countable neighborhood basis. The argument involves a classification of compactifications, the lemma that Ind of a space cannot be increased by adjoining a single point, and the observation of Yu. M. Smirnov that these spaces R have the property that $B-R$ has the Lindelöf property for any compactification B, so that $\mathrm{Ind}(B-R)=\mathrm{ind}(B-R)$. *J. Isbell* (Seattle, Wash.)

4254:
Aleksandrov, P.; and Ponomarev, V. On bicompact extensions of topological spaces. Dokl. Akad. Nauk SSSR **121** (1958), 575–578. (Russian)

A subset N of a space B is said to be zero-dimensionally embedded in B provided every point of B has a basis of neighborhoods whose boundaries are disjoint from N. The authors show that semibicompact spaces R are characterized by the property that there exists a compactification B of R in which $B-R$ is zero-dimensionally embedded. Moreover, if Ind $N=0$, then N is zero-dimensionally embedded in any compactification. In particular, if R has compactification B with $\mathrm{Ind}(B-R)=0$ then R is semibicompact. The converse apparently remains an open problem.

The paper includes, and employs, a classification of compactifications somewhat different from Sklyarenko's [cf. preceding review]. *J. Isbell* (Seattle, Wash.)

4255:
Fomin, S. V. On the connection between proximity spaces and the bicompact extensions of completely regular spaces. Dokl. Akad. Nauk SSSR **121** (1958), 236–238. (Russian)

The author shows that the bounded Δ-continuous functions on a proximity space form a ring, which in turn induces a coarser proximity structure on the given space; however, it is the same structure if the usual definition of a proximity space is used.
J. Isbell (Seattle, Wash.)

4256:
Nagata, Jun-iti. A contribution to the theory of metrization. J. Inst. Polytech. Osaka City Univ. Ser. A **8** (1957), 185–192.

An expanded version of the author's note [Proc. Japan Acad. **33** (1957), 128–130; MR **19**, 157]. The principal theorem and its proof remain the same. Some more applications are given, with proofs, showing how this theorem easily implies the sufficiency of just about every previously known characterization of metrizability. It is also shown that condition (b) of the theorem can be dropped if the space is assumed to be topologically complete (i.e., a G_δ in a compact Hausdorff space), but not in general. *E. Michael* (Seattle, Wash.)

4257:
Gutiérrez-Burzaco, Mario. Extension of uniform homotopies. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 61–69.

The author considers the category of spaces with a uniform structure (especially metric spaces) and uniformly continuous mappings. Appropriate definitions are given for notions like uniform neighborhood extension property (u-NEP), uniform absolute neighborhood retract (u-ANR) and uniform homotopy extension property. Ten theorems are proved relating these notions; they are mostly natural analogues of propositions which are well known in the case of the category of metrizable spaces and continuous maps. Sample theorem: a bounded metric space X is a u-ANR if and only if every closed subset S_0 of an arbitrary metric space S has the u-NEP in S with respect to X. *S. Mardešić* (Princeton, N.J.)

4258:
Marr, J. M. On spaces which are not of countable character. Proc. Amer. Math. Soc. **9** (1958), 780–781.

Let X be a connected Hausdorff space, and let A be a nonempty, proper closed set in X. 1. If X is compact, locally connected, and has a countable base, then there exists a continuous $f:X\to X$ such that $f(X\setminus A)$ meets A. 2. If A consists of all points that are not limit points of countable sets, and if each point of $X\setminus A$ has a countable base, then for every continuous $f:X\to X$, $f(X\setminus A)$ is disjoint from A; hence if now $A=\{x_0\}$ and $x_0\in f(X)$, then $f(x_0)=x_0$. *L. Gillman* (Princeton, N.J.)

4259:

Bielecki, A.; et Kisyński, J. Une remarque à propos de deux notes de Z. Szmydt. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 15–17.

The authors prove the following fixed point theorem (suggested by one applied by Z. Szmydt [Bull. Acad. Polon. Sci. Cl. III **4** (1956), 579–584; MR **18**, 741] to partial differential equations): For $i=1, 2$, let E_i be complete metric spaces with elements $u_i, v_i, x_i, y_i, \cdots$ and metrics $d_i(x_i, y_i)$. Let $u_i=\varphi_i(x_1, x_2)$ be a map of $E_1\times E_2$ into E_i such that $d_i(u_i, v_i)\leqq a_{i1}d_1(x_1, y_1)+a_{i2}d_2(x_2, y_2)$ for all x_1, x_2, y_1, y_2, where $v_i=\varphi_i(y_1, y_2)$ and a_{ij} are positive constants satisfying $a_{11}<1$, $a_{22}<1$, $a_{12}a_{21}<(1-a_{11})(1-a_{22})$. Then the map T: $(x_1, x_2)\rightarrow(\varphi_1, \varphi_2)$ of $E_1\times E_2$ into $E_1\times E_2$ has a unique fixed point. The proof, given in a few lines, is obtained by showing that the map T is a contraction if a metric of the form $d((x_1, x_2), (y_1, y_2))=\alpha d_1(x_1, y_1)+\beta d_2(x_2, y_2)$, where α, β are suitable constants, is introduced on $E_1\times E_2$.

P. Hartman (Baltimore, Md.)

ALGEBRAIC TOPOLOGY

See also 3929, 4203, 4204, 4431, 4432.

4260:

Markov, A. The insolubility of the problem of homeomorphy. Dokl. Akad. Nauk SSSR **121** (1958), 218–220. (Russian)

The problem of homeomorphy is: given the simplexes and incidences of two finite polyhedra, to determine if they are homeomorphic. The author announces an effective construction assigning to each finite system S of generators and relations for a group a 4-manifold $M(S)$ having that group as fundamental group in such a way that $M(S)$ and $M(T)$ will be homeomorphic if (and only if) they have the same fundamental group. Now from S. I. Adyan's proof of the algorithmic insolubility of the problem of determining whether S determines a nonzero group [same Dokl. **103** (1955), 533–535; MR **18**,455], it follows that the problem of homeomorphy is insoluble.

J. Isbell (Seattle, Wash.)

4261:

Isbell, J. R. Euclidean and weak uniformities. Pacific J. Math. **8** (1958), 67–86; erratum, 941.

If X is a simplicial complex then μX (called uniform complex) is the set of points $x=(x_\alpha)$ of X provided with the uniformity induced by the distance $d(x, y)=\max|x_\alpha-y_\alpha|$; if μX is uniformly equivalent with some SCE^n, then μX (and X) is called a Euclidean complex; a covering of a space is called Euclidean if its nerve is a Euclidean complex. Various results involving these notions, as well as properties of coverings such as star-boundedness, are contained in sections 1 and 2. A characterization of Euclidean complexes by means of mappings of the set of vertices into E^n is given. It is pointed out that, as a consequence, Euclidean complexes coincide, essentially, with Smirnov's [Mat. Sb. N.S. **40(82)** (1956), 137–156; MR **19**, 300] uniform (geometrical) complexes. Countable star-finite finite-dimensional complexes X are characterized by various properties of μX and are shown to be essentially equivalent with Smirnov's [loc. cit.] Lebesgue (geometrical) complexes. A simple necessary and sufficient condition for a uniform space to be uniformly equivalent with an open interval of R (the reals) is given, as well as a more complicated one for the uniform equivalence with a subset of R. In section 3 there is defined, for any weak (i.e., induced by a set of real-valued functions) uniformity μ on a set X, the "weak derivative" $w\mu$, which is shown to be precisely the uniformity induced by functions $g(f_1, \cdots, f_n)$, g continuous on E^n, f_i uniformly continuous on μX. It is proved that μX is uniformly equivalent with a (closed) subset of E^n, where E^n is provided with the finest uniformity consistent with its (usual) topology, if and only if μ is weak, $w\mu=\mu$, and there is a uniform covering of μX consisting of precompact (compact) metrizable spaces with completions of bounded dimension. {Reviewer's remark: as pointed out by the author, "finer than" is to be replaced by "equal to" in 3.10; the original statement is false.} *M. Katětov* (Prague)

4262:

Bauer, Friedrich Wilhelm. Über Fortsetzungen von Homologiestrukturen. Math. Ann. **135** (1958), 93–114.

The classical Alexander duality theorem states that the cohomology groups of a compact subset of an n-sphere, S^n, and the homology groups of the complement are isomorphic (in the complementary dimensions). This has been extended to a duality theorem relating the homology (or cohomology) properties of an arbitrary subset of S^n to that of its complement by S. Kaplan [Trans. Amer. Math. Soc. **62** (1947), 248–271; MR **9**, 456], P. Alexandrov, and K. A. Sitnikov [for references to the series of papers by Alexandrov and Sitnikov, see MR **17**, 69, 70, 118, 1120]. The papers of these three authors are rather involved and complicated. Apparently one of the goals of the present paper is to put this work on a more conceptual basis so it can be more easily comprehended.

The author proceeds as follows. The set of all compact subsets of S^n is a lattice; if the inclusion maps are also considered, it is a category A_1 in the sense of Eilenberg and MacLane [Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 537–543; MR **4**, 134]. The set of all complements of compact subsets of S^n is similarly a lattice and category, B_1. Cohomology theory is a functor from the category A_1 to the category of abelian groups, and homology theory is a functor from B_1 to the category of abelian groups. The Alexander duality theorem is an isomorphism between these two functors. The problem of extending the Alexander duality theorem is a problem of suitably extending cohomology and homology theory to categories larger than A_1 and B_1, and then extending the known isomorphism to the extended functors.

The author considers abstract problems of this type, concerning the extension of functors defined on categories which are also lattices, and discusses the possibility of extending isomorphisms between such functors. In his work he makes use of certain lattice theoretic notions, such as the completion of a lattice, and the concept of an "atom" of a partially ordered set. The principal results are much too involved to state here. At the end of the paper there is a brief discussion of applications; the author promises to give further consideration to applications in a future paper.

W. S. Massey (Providence, R.I.)

4263:

Bokštein, M. Tensor products of systems of groups and universal coefficient theorems for homologies and cohomologies. Dokl. Akad. Nauk SSSR **119** (1958), 1066–1069. (Russian)

Let X be a Hausdorff compact space, G an abelian group, $H^q(X; G)$ the Čech cohomology group and $H_m{}^q(X)=$

$H^q(X, Z_m)$, $m=0, 1, 2, \cdots$. Then there exists a well-known [e.g., see F. P. Peterson, Amer. J. Math. **78** (1956), 243–258; MR **18**, 919] exact sequence

$$0\to H_0^q(X)\otimes G\to H^q(X, G)\to \mathrm{Tor}(H_0^{q+1}(X), G)\to 0.$$

In this paper it is proved that the sequence splits (this was known only in the case of metrizable compacta [S. Eilenberg and S. MacLane, Ann. of Math. (2) **43** (1942), 757–831; MR **4**, 88].

The author first gives an explicit description of the group $H^q(X, G)$ in terms of the modular spectrum of X, i.e., groups $H_m^q(X)$ and maps $\pi_m^{m'}:H_{m'}^q(X)\to H_m^q(X)$, $\omega_{m'}^{m}:H_m^q(X)\to H_{m'}^q(X)$, which are induced by obvious maps $Z_{m'}\to Z_m$, $Z_m\to Z_{m'}$; here m is a divisor of m'. The result is actually a reformulation of an older theorem due to the author [Dokl. Akad. Nauk SSSR **37** (1942), 243–245; MR **5**, 48], which, however, became much neater through the use of exact sequences and a notion of tensor product of two systems of groups with mappings. Next it is shown (by means of the first theorem of Prüffer [see e.g. A. G. Kurosh, The theory of groups, Vol. I, Chelsea, New York, 1955; MR **17**, 124]) that the sequences

$$0\to H_0^q(X)/mH_0^q(X)\to H_m^q(X)\to {}_m[H_0^{q+1}(X)]\to 0$$

(special cases of the sequences of above) split. The splitting maps from the third term into the second are chosen in a special way so as to guarantee a certain compatibility with respect to transitions between various m. This combined with the explicit description of $H^q(X, G)$ yields the wanted result. Same result holds for relative cohomology of compact pairs.

S. Mardešić (Princeton, N.J.)

4264:

Brahana, Thomas R. Axioms for local homology theory. Duke Math. J. **25** (1958), 381–399.

Various definitions of "local homology group at a point" have been given in the literature, but no attempt has previously been made to make a "homology theory" of these groups in a functorial way. The author gives a system of axioms analogous to the Eilenberg-Steenrod axioms, in which he replaces the global groups $H_q(X, A)$ of a pair by the groups $L_q(x_0:X, A)$ of a "local pair". The main difficulty, of course, is with the homotopy axiom, and this must be replaced by one using a more complicated concept of "local homotopy". The resulting axioms are shown to be satisfied when $(x_0:X, A)$ is "locally triangulable", and a uniqueness theorem is proved for the category of locally triangulable local pairs. A local simplicial approximation theorem has first to be established, and the main theorem is then proved in stages, analogous to those of the global proof of Eilenberg-Steenrod; but the definitions and details must naturally be more complicated than with the global case. Finally, the global theory is embedded in that of the author, by inserting $H_q(X, A)$ for $L_{q+1}(x_0: \hat{X}, \hat{A})$, where $\hat{X}$ denotes the cone over X, of vertex x_0. Thus, the global uniqueness theorem follows from the local.

H. B. Griffiths (Bristol)

4265:

Wilder, R. L. Monotone mappings of manifolds. II. Michigan Math. J. **5** (1958), 19–23.

[Part I: Pacific J. Math. **7** (1957), 1519–1528; MR **19**, 1188.] The author proves the following theorem. Let $f:S\to S'$ be an $(n-1)$-monotone, proper mapping of an n-gm S onto an at most n-dimensional, locally compact Hausdorff space S' of more than one point. Suppose that for each $x'\in S'$, the set $f^{-1}x'$ lies in an orientable n-gm in S. Then S' is a locally orientable n-gm of the same homology type as S. Here, "n-gm" means "n-dimensional generalised manifold", defined (with the monotone condition) using homology over a field; and the map f is proper if f^{-1} (compact set)=compact set. The above result extends the author's earlier version concerning compact n-gms. Two simple examples are given, showing that if f is not proper, or if $f^{-1}x'$ does not satisfy the orientability condition, then the theorem does not hold. (For the second, let f be the mapping of the product of the projective plane P^2 and the circle, obtained by shrinking one fibre P^2 to a point, and take coefficients mod 3.) Finally, a second version of the theorem is given, with S' assumed to be finite-dimensional, and f monotone over the integers. *H. B. Griffiths* (Bristol)

4266:

de Lyra, C. B. On the homotopy type of a factor space. An. Acad. Brasil. Ci. **30** (1958), 37–41.

In 1949, Eckmann, Samelson, and G. W. Whitehead proved the following theorem: A fibration of the n-sphere with a separable metric base space and an s-dimensional torus as fibre exists if and only if n is odd and $s=1$ [Bull. Amer. Math. Soc. **55** (1949), 433–438; MR **10**, 728]. In the present paper, the author points out that the hypothesis that the base space is separable metric is unnecessary; it follows from the hypothesis that we have a locally trivial fibration with compact fibres. He also points out that the base space must be a finite dimensional ANR having the same homotopy type as a complex projective space. {Reviewer's note: The author repeatedly lists the authors of the 1949 paper mentioned above as "Borel, Eckmann, and Samelson".}

W. S. Massey (Providence, R.I.)

4267:

Bucur, I. Une nouvelle démonstration des formules de dualité des classes de Chern. Rev. Math. Pures Appl. **2** (1957), 419–422.

The proof given here that the two definitions of Chern classes (as obstructions, c_{2i}, and as symmetric functions, C_{2i}) coincide, does not make use of the duality formula for the c_{2i}, in contrast to the procedure of A. Borel and J.-P. Serre [Amer. J. Math. **75** (1953), 409–448; MR **15**, 338]. It is first shown that the c_{2i} agree with the C_{2i}, up to sign. By considering products of complex manifolds and the Euler characteristic, one shows inductively that $c_{2n}=C_{2n}$ for $U(n)$-bundles. Using this, one shows $c_{2i}=C_{2i}$ for the complex projective spaces, and therefore generally. The duality formula for the c_{2i} follows.

H. Samelson (Ann Arbor, Mich.)

4268:

Liu, Ya-shing. On ring-like spaces. I. Advancement in Math. **3** (1957), 404–408. (Chinese. English summary)

Let R be a space with continuous addition, inversion and multiplication which satisfies the ring axioms up to homotopy. Such a space is called a ring-like space. The first theorem in this paper states that the property of being a ring-like space is a restricted homotopy invariant; that is to say, if X is a ring-like space with x_0 as the homotopy zero element and if (Y, y_0) is a pair which is homotopically equivalent to the pair (X, x_0), then Y is also a ring-like space with y_0 as the homotopy zero element. The second theorem states that the topological product of two ring-like spaces is a ring-like space.

Sze-tsen Hu (Detroit, Mich.)

4269:

Weier, Joseph. Ueber Probleme aus der Topologie der Ebene und der Flächen. Math. Japon. **4** (1956), 101–105.

Let f be an essential mapping from a planar domain into a circumference S. It is shown that the dimension of the set $(f)^{-1}(p)$ is at least one for every point p in S. — A construction is given for a two-dimensional bounded finite euclidean manifold N and a continuous mapping g from N into itself so that the number of fixed point classes of g is equal to one, yet every mapping from N into itself which is homotopic to g has at least two fixed points. This example shows that a conjecture of J. Nielsen is false. — For plane vector fields a concept of multiplicity is defined, in terms of which there is stated a sufficient condition in order that two fields belong to the same homotopy class.

P. V. Reichelderfer (Columbus, Ohio)

4270:

Weier, Joseph. Untersuchungen über Abbildung von Sphären. Math. Japon. **4** (1957), 179–205.

Some known results and some new results concerning continuous mappings from special subsets of euclidean space are established. Examples follow. Let S be an n-sphere of unit radius. Denote by F the class of all continuous mappings f from S into itself which have the property that the distance between p and $f(p)$ is greater than zero but less than one for every point p in S. Two mappings in F are said to belong to the same class if there is a homotopy of mappings connecting them in F. A new proof is offered for the known fact that if $n \geqq 5$ then the number of classes in F does not exceed two. — Let A be a 2-simplex in a 3-sphere S, and q a point in a 2-sphere T. Then if f is a continuous mapping from S into T it is shown that there exists a mapping f' from S into T which is homotopic to f and such that $(f')^{-1}(q)=\bar{A}-A$. If the degree of $f'|\bar{A}$ is zero, then f is nonessential. — For $n \geqq 5$ let S be an n-sphere and T an $(n-1)$-sphere. Let (f, g), (f', g') be two pairs of mappings from S into T which are homotopic and such that there is a single point a in S which is the only point of S where f and g have the same value and also the only point of S where f' and g' have the same value. It is shown that there is a homotopy of pairs (f^τ, g^τ) of mappings from S into T connecting (f, g) and (f', g') such that for each value of τ the point a is the only point of S where f^τ and g^τ have the same value.

P. V. Reichelderfer (Columbus, Ohio)

4271:

Weier, Joseph. Un théorème d'intersection. Rend. Circ. Mat. Palermo (2) **6** (1957), 343–348.

Let P and Q be compact orientable topological manifolds with $\dim P = 1+\dim Q > 2$, and let $f\colon P \to Q$ be a mapping. Then there exist maps $g, g'\colon P \to Q$, homotopic to f and such that $S=\{x|x \in P, g(x)=g'(x)\}$ is either empty or a finite collection of mutually disjoint circles. The author derives this proposition from three others, which are given without proof. The first of these propositions asserts (under the assumptions above) the existence of g and g' with S being either empty or a compact 1-dimensional polyhedron having no isolated points.

S. Mardešić (Princeton, N.J.)

4272:

Čulík, Karel. Zur Theorie der Graphen. Časopis Pěst. Mat. **83** (1958), 133–155. (Czech and Russian summaries)

The author deals with graph theory from an exclusively algebraic point of view, and is predominantly concerned with terminology. A graph $F(f)$ is a set F together with a binary relation $\rho \subset F\times F$, and f is the characteristic function of the set ρ. A large number of graph-theoretical concepts, both old ones and new ones, are defined. Among the new ones the most important are: homomorphism, quotient graph, simplicity, and cardinal product. Given graphs $F(f)$ and $G(g)$, a function ϕ of F onto G is a homomorphism if $f(x, y)=g(\phi(x), \phi(y))$ for all $x, y \in F$. Let $\sim$ be an equivalence relation on F such that $x\sim x'$, $y\sim y'$ implies $f(x, y)=f(x', y')$; then the graph $G(g)$ given by $G=F/\sim$, $g(X, Y)=f(x, y)$, where $x \in X$, $y \in Y$ and $X, Y \in G$, is a quotient graph of $F(f)$. Every homomorphism gives rise to a quotient graph, and vice versa. A graph $F(f)$ is simple if every homomorphic image of $F(f)$ is isomorphic to $F(f)$. A necessary and sufficient condition that $F(f)$ be simple is that for any two distinct $x, y \in F$ there is a $z \in F$ with $f(x, z)\neq f(y, z)$ or $f(z, x)\neq f(z, y)$. Every graph has a unique simple quotient graph. Homomorphic images and pre-images of connected graphs are connected. If $\{F_\lambda(f_\lambda)|\lambda \in \Lambda\}$ is a family of graphs, their cardinal product $F(f)=\prod F_\lambda(f_\lambda)$ is given by $F=\prod F_\lambda$ and $f(x, y)=\prod f_\lambda(p_\lambda x, p_\lambda y)$, where p_λ is the usual projection of F onto F_λ. Decomposition of graphs into cardinally indecomposable factors is not unique (not even for finite graphs). Another property is that the product of connected graphs is not necessarily connected.

G. Sabidussi (New Orleans, La.)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 3942, 4004, 4203, 4204, 4364.

4273:

✱Strubecker, Karl. Differentialgeometrie. II. Theorie der Flächenmetrik. Sammlung Göschen Bd. 1179/1179a. Walter de Gruyter & Co., Berlin, 1958. 195 pp. DM 4.80.

[For vol. I see MR **16**, 954]. The work will consist of three volumes, of which the third will deal with the Gaussian curvature of surfaces and with their extrinsic geometry (depending on the space in which they are imbedded). The present volume contains a detailed treatment of their intrinsic geometry based upon their metric. There are four sections: surface-metric; vector analysis on surfaces; mapping of one surface on another; geodesic curvature, geodesic curves, absolute parallelism. In the first section complex differential geometry (isotropic curves, etc.) is given considerable attention. The second section deals with the two Beltrami differential operators, divergence, curl, and Green's formulas. The third section deals in detail with conformal mapping of a surface on the plane, illustrated in particular by three maps of the surface of the Earth. The fourth section culminates in a treatment of absolute differentiation along a curve on a surface, with Frenet's formulas.

4274:

Bakel'man, I. Ya. Non-regular surfaces of bounded external curvature. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 631–632. (Russian)

The paper defines a class of surfaces F in E^3 which have "locally bounded extrinsic curvature" and comprise the convex surfaces, the surfaces representable in the form $z=f(x, y)$, where $f(x, y)$ is the difference of two convex functions, and also the surfaces of locally bounded intrinsic curvature of A. D. Alexandrov. Every point $p \in F$ has a neighborhood $\mathscr{U}$ representable in the form $z=f(x, y)$. A tangent plane to the contingens of F at a point

of $\mathscr{U}$ is not normal to $z=0$. If $p_i \to p$; $p_i, p \in F$, then the limit of a converging sequence of tangent planes of the contingenses of F at p_i is a tangent plane of the contingens of F at p.

Let C be an open set in $\mathscr{U}$, with $\bar{C} \subset \mathscr{U}$, and let $M_1, \cdots, M_n$ be disjoint open sets in C whose projections on $z=0$ are convex. Represent the boundary of the convex closure of M_i in the form $M_i' \cup M_i''$ where M_i' is convex relative to $z=0$ and M_i'' is concave, and let σ_i', σ_i'' be the measures of the spherical images of M_i' and M_i''. F has locally bounded extrinsic curvature if $\sup_{(M_i)} \sum_{i=1}^{n} (\sigma_i' + \sigma_i'')$ is finite for every $C \subset \mathscr{U} \subset F$.

F can then be locally, uniformly approximated by regular surfaces which have uniformly bounded positive part of their integral curvatures and whose intrinsic metrics tend to that of F. *H. Busemann* (Cambridge, Mass.)

4275:

Sypták, M. Les spirales logarithmiques (exponentielles) et coniques dans l'espace euclidien à p dimension. Acta Fac. Nat. Univ. Comenian. Math. 2 (1958), 177–186. (Czech. Russian and French summaries)

The curves in euclidean p-space with natural equations $k_i s = A_i$, k_i the curvatures and A_i constants, $i=1, 2, \cdots, p-1$, are logarithmic spirals for $p=2$ and conical spirals for $p=3$. It is shown that the equations of the curves for $p=2n$ can be written in the form $x_{2i-1} = r_i s \cos(l_i \log s)$, $x_{2i} = r_i s \sin(l_i \log s)$, $i=1, 2, \cdots, n$; for $p=2n+1$ we must add $x_{2n+1} = Cs$. The laws of formation of the constants r_i, l_i, C are derived, as well as certain properties of the curves.

D. J. Struik (Cambridge, Mass.)

4276:

Vincensini, Paul. Sur un invariant du groupe des équivalences superficielles de l'espace euclidien à trois dimensions. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 23 (1957), 217–220.

With every point P of a surface S with equation $\bar{P} = \bar{P}(u, v)$ in three-space is associated a point I in the tangent plane π by means of the equation $\bar{I} = \bar{P} + a\bar{P}_u + b\bar{P}_v$, where a and b are functions of u and v. If, when P describes a curve C on S, the locus of I corresponds to Γ by means of orthogonality of corresponding elements, we speak of the rule of association (P, I). In this case Γ belongs to a net $A\,du^2 + B\,du\,dv + C\,dv^2 = 0$, where $A = E(1+a_u) + Fb_u + \frac{1}{2}aE_u + \frac{1}{2}bE_v$, etc., so that for a given rule of association (P, I) this net is invariant under deformations of S. If the net is orthogonal, $a_u + b_v + aH_u + bH_v + 2 = 0$, $H = (EF - G^2)^{\frac{1}{2}}$, so that now the rules of association (P, I) are also invariant with respect to equiareal transformations of S. If we take a surface S' obtained from S by an equiareal transformation, and two point pairs (P, I) on S, (P', I') on S' associated by a rule (a, b), then I and I' correspond in a unimodular affinity. Every equiareal transformation of surfaces extended in the above-mentioned way to the rules of association (P, I), transforms every rule (P, I) with respect to a surface S for which the invariant net is orthogonal into a rule (P', I') with respect to a transformed surface S' for which the invariant net is also orthogonal. *D. J. Struik* (Cambridge, Mass.)

4277:

Löbell, Frank. Der Einfluss einer Flächentransformation auf die geodätischen Krümmungen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. 1957, 15–24 (1958).

A. Marussi [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) 16 (1954), 478–483; MR 16, 745] and M. Mineo [Matematiche, Catania 11 (1956), 1–7; MR 18, 229], among others, have discussed this problem recently. Let surface F be mapped onto surface $\bar{F}$. Select corresponding real orthogonal networks as parameter curves on F and $\bar{F}$. Let a curve on F make an angle φ with the first family of the network. Let $m = d\bar{s}/ds$ for the direction φ; let m_1 and m_2 be the values of m for the parameter directions. Then if g and $\bar{g}$ are the geodesic curvatures of the curve and its image, the author works straightforwardly from the Liouville formula to obtain $\bar{g} = (m_1 m_2 g + \Phi)/m^3$, where Φ is a cubic form in $\cos\varphi$ and $\sin\varphi$ with coefficients which are functions of position on F. From this formula many conclusions follow in simple fashion. As an example, if Φ vanishes identically we have Dini's case, where the geodesics of F and $\bar{F}$ correspond. If Φ does not vanish identically, there are three surface directions for which geodesics correspond to geodesics. The cylinder projections of the sphere are presented as illustrations. Finally, it is noted that the change in the Gaussian curvature under an area-preserving transformation can be computed, starting from Bonnet's formula for K.

A. Schwartz (New York, N.Y.)

4278:

Grincevičyus, K. I. A linear complex associated with a differentiable neighborhood of second order of a ray of the complex. Uspehi Mat. Nauk (N.S.) 13 (1958), no. 2(80), 175–180. (Russian)

Les repères locaux d'un complexe K de droites $[A_1 A_2]$ dans S_3 soient particularisés de sorte que

$$\omega_2^3 = -\omega_1^4,$$

$$\omega_3^4 - \omega_2^1 = h_{3333}\omega_1^3 + 2h_{3334}\omega_1^4 + (h - h_{3344})\omega_2^4,$$

$$\omega_2^3 - \omega_1^1 - \omega_3^3 + \omega_4^4 = 2h_{3334}\omega_1^3 + 2(h + 2h_{3344})\omega_1^4 - 2h_{3444}\omega_2^4,$$

$$\omega_4^3 - \omega_1^2 = (h - h_{3344})\omega_1^3 - 2h_{3444}\omega_1^4 + h_{4444}\omega_2^4.$$

Soit L une congruence arbitraire de K qui passe par $l \in K$; si la transformée de Laplace l_1 passe par le point $A = \xi^i A_i$, l_{-1} est située sur le cône quadratique $\mathfrak{X}$ ayant pour sommet $M = \xi^4 A_1 + \xi^3 A_2$. Soient $Y_i = \eta_i^p A_p$ $(i, p = 1, 2)$ les points qui correspondent au point M par la relation $h_{\alpha\beta\gamma\varepsilon}\xi^{5-\alpha}\xi^{5-\beta}\eta^{5-\gamma}\eta^{5-\varepsilon} = 0$. Le plan tangent le long de la droite l du cône des droites du complexe qui passe par Y_i coupe $\mathfrak{X}$ en droites l et k_i; la transformation $A \to (k_1, k_2)$ donne naissance au complexe linéaire $p^{13} + p^{42} - hp^{34} = 0$.

A. Švec (Prague)

4279:

Geĭdel'man, R. M. Stratification of congruences of circles and spheres. Mat. Sb. N.S. 43(85) (1957), 295–322. (Russian)

The concept of stratification can be extended to families of subspaces (C_1) and (C_2) in a space with a fundamental Lie group. Then we can say, in general, that when a one to one correspondence between (C_1) and (C_2) is established such that the elements of a subspace C_1 can, in an appropriate way, be included in manifolds of which the differential neighborhood is incident with the corresponding subspace C_2, the family (C_1) stratifies the family (C_2). If (C_2) also stratifies family (C_1) we have double stratification. This definition, applied to two congruences of circles (C_1) and (C_2) in conformal three-space, leads to the following form of stratification. If to (C_1) we can adjoin ∞^1 two-parametric families of spheres $(S_1, S_0$; C_1 is given as intersection of spheres S_1 and $S_0)$ such that the circle C_2 passes through the characteristic points of all spheres $S_1 + kS_0$, then the congruence (C_1) stratifies

the congruence (C_2). This stratification is systematically investigated with the aid of pentaspherical coordinates. Then, in the second section of the paper, the case is studied of the stratification of k-parametric congruences of $(n-k)$-dimensional spheres (C_1), (C_2) in n-dimensional conformal space $(n \geqq 2k-2)$. Here we adjoin to (C_1) ∞^{k-1} k-parametric families of hyperspheres (S) passing through the sphere C_1 of family (C_1) such that their characteristics are on the corresponding sphere C_2.

D. J. Struik (Cambridge, Mass.)

4280:

Šulikovskiĭ, V. I. Tensor methods in the theory of congruences. Kazan. Gos. Univ. Uč. Zap. **112** (1952), no. 10, 57–76. (Russian)

4281:

✱Wills, A. P. Vector analysis with an introduction to tensor analysis. Dover Publications, Inc., New York, 1958. xxxii+285 pp. $1.75.

This book is an unabridged corrected republication of the 1931 edition, published by Prentiss-Hall, New York. It contains an exposition of such topics as basic vector algebra and vector calculus, scalar and vector fields, relations of volume, surface and line integrals, scalar and vector potential functions, linear vector functions and dyadics, transformation theory, non-Euclidean manifolds and tensor theory, together with a short historical introduction. Problems and a few applications are included.

4282:

Gołąb, S.; et Pidek-Łopuszańska, H. Sur l'algèbre des objets géométriques de première classe à une composante. Ann. Polon. Math. **4** (1958), 226–248.

The purpose of this paper is to determine all binary operations on geometric objects (g.o.) of one component, of type Δ, yielding again such objects. Every g.o. of type Δ, with one component, is equivalent to: (1) a scalar; (2) a bi-scalar (is properly transformed only when $\Delta<0$: $\omega \to \nu(\omega)$, with $\nu[\nu(\omega)]=\omega$); (3) a Weyl density: $\omega \to |\Delta|\omega$; (4) a density: $\omega \to \Delta\omega$. The types are denoted by S, B, W, G respectively. Let ω_1 and ω_2 be of any of these types (or equivalent to them), and let $\omega_3=f(\omega_1, \omega_2)$ also be one of these, for suitable f. It is shown that f can be found only for the combinations of types of $(\omega_1, \omega_2, \omega_3)$ in which ω_3 is not "higher" in type than ω_1 and ω_2 under the partial ordering S<B<G, S<W<G, except that the triple (B, W, G) is also permissible. These facts are simple consequences of the homomorphisms and decomposition of the underlying groups. Since the operation is assumed to be truly binary, certain triples are lacking, such as (S, G, S), (B, G, B), (S, W, S), (B, W, S), (B, W, B). In each case all permissible f have been characterized. The paper concludes with a discussion of multiplication and addition of various kinds of densities.

{Reviewer's note: It seems that the proof of the case (B, B, B) (end of § 3) is not complete, since there may be f, ω_1, ω_2, ω_1^*, ω_2^* such that $f(\omega_1, \omega_2)=f(\omega_1^*, \omega_2^*)$ and $f(\bar{\omega}_1, \bar{\omega}_2) \neq f(\bar{\omega}_1^*, \bar{\omega}_2^*)$, in which case there is no function ν satisfying $f(\bar{\omega}_1, \bar{\omega}_2)=\nu[f(\omega_1, \omega_2)]$.}

A. Nijenhuis (Seattle, Wash.)

4283:

Gołąb, S.; und Kucharzewski, M. Über die Invarianz gewisser Eigenschaften von Affinoren bei Transformationen der entsprechenden Untergruppen der allgemeinen affinen Gruppe. Tensor (N.S.) **8** (1958), 1–7.

It is well known that the property of being a geometric object depends upon the choice of the transformation group. Here it is shown that certain non-geometric objects become geometric objects when a subgroup of the given group is considered. The principal theorem is that the property of symmetry $a_j{}^i=a_i{}^j$ for a mixed tensor is invariant only for the similarity subgroup of the affine group.

C. B. Allendoerfer (Seattle, Wash.)

4284:

Pâquet, P. V. Sur la dérivée des intégrales de variétés. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 107–125.

Poursuivant des préoccupations didactiques, l'auteur dégage le contenu intuitif de la formule

$$D_v\int_{\sigma^n}\Omega_n=\int_{\sigma^n}E_v d\Omega_n+\oint_{\partial\sigma^n}E_v\Omega_n,$$

forme globale de la formule locale $D_v\Omega_n=E_v d\Omega_n+dE_v\Omega_n$ qui lie la dérivée D_v conformément à un champ de vecteurs v, d'une forme différentielle extérieure Ω_n d'un espace E^N, à l'opération d de la différentiation extérieure et à l'opération E_v de la substitution confirmément au champ v. L'exposé est mené dans l'esprit de ce que l'auteur a dénommé jadis la méthode de Grassmann en géométrie différentielle. La dualité variété-covariété y est intuitivement dégagée. Pour une formalisation plus détaillée de celle-ci l'auteur renvoie au récent traité de Hassler Whitney sur la théorie de l'intégration géométrique [Geometric integration theory, Princeton Univ. Press, 1957; MR **19**, 309].

Résumé de l'auteur

4285:

Norden, A. P. Biaffine space and its mapping on itself. Kazan. Gos. Univ. Uč. Zap. **112** (1952), no. 10, 3–11. (Russian)

4286:

Norden, A. P. Theory of curves of biaffine space. Kazan. Gos. Univ. Uč. Zap. **112** (1952), no. 10, 13–26. (Russian)

4287:

Cypkin, M. E. Differential geometry of a ruled space and its applications to the theory of ruled surfaces. Kazan. Gos. Univ. Uč. Zap. **112** (1952), no. 10, 77–99. (Russian)

4288:

Cypkin, M. E. Application of Kotelnikov's transport principle to the theory of ruled surfaces. Kazan. Gos. Univ. Uč. Zap. **112** (1952), no. 10, 101–107. (Russian)

4289:

Arghiriade, E. Sur les quadriques de C. C. Hsiung. Rev. Math. Pures Appl. **2** (1957), 535–545.

Let t_u and t_v be the asymptotic tangents at a general point x of a curve C_λ on an analytic non-ruled surface S in a three-dimensional projective space. Associated with two points T, T^* on t_u, t_v, respectively, the reviewer [Duke Math. J. **11** (1944), 59–64; MR **5**, 217] has defined two quadrics H_u and H_v. In the present paper, the author gives the following new definition of the quadrics H_u and H_v, together with some new properties of these quadrics and some other convariant configurations. Let R_u (R_v) be the ruled surface of the asymptotic u- $(v$-$)$ tangents constructed at the points of C_λ. Then in the Davis-Gambier's pencil of quadrics, each of which has contact of the first order with R_u, R_v along t_u, t_v, respectively, the quadric having contact of the second order with R_u

at the point T is the quadric H_u. H_v can be defined in the same way by interchanging u and v.

C. C. Hsiung (Bethlehem, Pa.)

4290:

Mayer, O. Remarques sur les systèmes triplement conjugués. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) 3 (1957), 183–195. (Romanian. Russian and French summaries)

Certain properties are derived concerning triply conjugate systems described by points, lines or planes in projective P_n. A class of such systems is introduced depending on $3n+3$ arbitrary functions of one argument. Special attention is given to the "systèmes assemblés" of Guichard. *D. J. Struik* (Cambridge, Mass.)

4291:

Finikov, S. P. *W*-transformations of Cartan manifolds of special projective type. Mat. Sb. N.S. **43(85)** (1957), 169–186. (Russian)

These Cartan manifolds, introduced by E. Cartan [Bull. Soc. Math. France **47** (1919), 125–160; **48** (1920), 132–208], are p-dimensional manifolds in projective space with an osculating space of dimension $2p$ which are built up by p one-parametric families of $(p-1)$-dimensional Cartan manifolds which form p-conjugated systems in the sense of Darboux. However, not every such system of Darboux is a Cartan manifold. The object of the present paper is the study of such manifolds under transformations W. Two p-dimensional manifolds S_p, S_p' in projective space are related by such transformations if a one to one correspondence exists between points M, M' of S_p, S_p' such that line MM' belongs to the tangent subspaces of S_p, S_p' at M, M' and the systems of second fundamental forms are such that each form of one system can be linearly expressed in the forms of the other system. The equations of the correspondence of two Cartan manifolds under such transformations are set up and their integration is accomplished under certain conditions.

D. J. Struik (Cambridge, Mass.)

4292:

Pinl, M. Minimalflächen fester Gaussscher Krümmung. Math. Ann. **136** (1958), 34–40.

The author considers minimal surfaces in complex n-dimensional Euclidean (or Riemannian) spaces R_n (resp. V_n), $n\geqq 3$ being the complex dimension, a minimal surface being a surface for which the mean curvature vector vanishes. In the case of Euclidean space, he shows that no such surface can have constant Gaussian curvature different from zero. In the case of a Riemannian manifold, however, according to a theorem of Kommerell and Struik [Schouten and Struik, Einführung in die neueren Methoden der Differentialgeometrie, Bd. II, Groningen, Göttingen, 1938; p. 94] any surface may be imbedded as a minimal surface in a suitable Riemannian manifold, hence minimal surfaces of non-vanishing constant Gaussian curvature are possible. The author constructs an example in which the unit sphere with Gaussian curvature $K\equiv+1$ is imbedded as a minimal surface in a cylinder space of (complex) dimension three. Finally he shows that it is, in a certain sense, a translation surface (Schiebfläche).

W. M. Boothby (Evanston, Ill.)

4293:

Favard, J. Théorèmes de Meusnier pour les variétés immergées dans les espaces de Riemann. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1 (49)** (1957), 265–268.

Consider a Riemann space V^m imbedded in V^n $(n>m)$. Let C be a curve on V^m. Then by differentiation we can find the first, second, etc., normals to C in V^n. These have normal and tangential components relative to V^m. Associated with these is a series of differential forms of higher degree which lead to analogues of the theorem of Meusnier. *C. B. Allendoerfer* (Seattle, Wash.)

4294:

Tomonaga, Yasuro. On extension of homogeneous Riemann spaces. Sûgaku **8** (1956/57), 100–102. (Japanese)

Another explanation of the results stated in the paper by the author in J. Math. Soc. Japan **5** (1953), 59–64 [MR **15**, 828]. *A. Kawaguchi* (Sapporo)

4295:

Šulikovskiĭ, V. I. Vector fields on a surface. Kazan. Gos. Univ. Uč. Zap. **112** (1952), no. 10, 49–56. (Russian)

4296:

Abramovici, F. Sur les espaces à connexion affine avec un champ parallèle maximal. Bul. Inst. Politehn. Bucureşti **18** (1956), no. 3–4, 145–148. (Romanian. Russian and French summaries)

This paper establishes the existence of systems of congruences in an affine space of n dimensions with respect to which the curvature tensor takes the form $\gamma^a{}_{bcd}=0$, $b\neq b'$, where b' is fixed. Two theorems on the groups of motions are demonstrated. (A maximal parallel field depends on $n-1$ arbitrary constants.)

D. J. Struik (Cambridge, Mass.)

4297:

Mirodan, R. Une nouvelle géometrisation des équations aux derivées partielles, linéaires et homogènes. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957), no. 14, 35–39. (Romanian. French and Russian summaries)

Following the method of C. M. Cramlet [Ann. of Math. (2) **31** (1930), 134–150], an affine connection is defined with the aid of the differential equation

$$a^{p_1p_2\cdots p_q}\frac{\partial^q z}{\partial x^{p_1}\partial x^{p_2}\cdots\partial x^{p_q}}+a^{p_1p_2\cdots p_{q-1}}\frac{\partial^{q-1}z}{\partial x^{p_1}\cdots\partial x^{p_{q-1}}}+\cdots$$
$$+a^p\frac{\partial z}{\partial x^p}+az=0,$$

under transformations $z'=z$, $x'^i=x'^i(x^1,\ \cdots,\ x^n)$. For $q=3$ the invariant form is $a^{ijk}D_{ijk}{}^3z+A^{ij}D_{ij}{}^2z+A^iD_iz+az=0$, where $D_ju_i=\partial u_i/\partial x^j+\Gamma_{ij}{}^vu_v$, the $\Gamma_{jk}{}^i$ are formed with the aid of the a^{ijk}, $A^{ij}=a^{ij}-3\Gamma_{vs}{}^ia^{vsj}$, $A^i=a^i-a^{vs}\Gamma_{vs}{}^i-a^{vst}(\partial\Gamma_{vs}{}^i/\partial x^t)+a^{vst}\Gamma_{vs}{}^u\Gamma_{ut}{}^i$. [See, for $n=2$, G. Vranceanu, Mathematica **20** (1944), 98–112; MR **6**, 229; N. Théodoresco, Bull. Math. Soc. Roumaine Sci. **41** (1939), 101–109; **42** (1940), 79–89; **43** (1941), 59–68; **44** (1942), 71–84; MR **7**, 80, 81 (and, by the way, also D. J. Struik and N. Wiener, C. R. Acad. Sci. Paris **185** (1927), 42–55; J. Math. Phys. **7** (1927), 1–23)].

D. J. Struik (Cambridge, Mass.)

4298:

Moór, Arthur. Untersuchungen in Räumen mit rekurrenter Krümmung. J. Reine Angew. Math. **199** (1958), 91–99.

The theory of Riemannian spaces of recurrent curvature $R_i{}^j{}_{kl,m}=\kappa_mR_i{}^j{}_{kl}$ was studied by H. S. Ruse [Proc. London Math. Soc. (2) **53** (1951), 212–229; MR **13**, 280] and A. G. Walker [Proc. London Math. Soc. (2) **52** (1950), 36–64; MR **12**, 283] in detail. In the present paper the analogous theory is given for affinely connected spaces and

for those metric spaces for which the connection parameters $L_j{}^i{}_k$ are non-symmetric. The necessary and sufficient condition for metrisability of an affinely connected space of recurrent curvature is found in the form that $g_{il}B_j{}^l{}_{kl}+g_{jl}B_i{}^l{}_{kl}=0$ have solutions g_{ij}, where $B_j{}^i{}_{kl}$ is the curvature tensor formed by $\Gamma_j{}^i{}_k=L_{(jk)}{}^i$, and from this fact several theorems are deduced. It is proved that if a space V_n of recurrent curvature is reduced to the product $V_r\times V_{n-r}$, then one of the component spaces must be flat. In the last chapter the curvature tensors of the general metric point spaces are discussed; for example, if there exist $n-1$ linearly independent parallel vector fields, then the curvature tensor vanishes identically.

A. Kawaguchi (Sapporo)

4299:

Willmore, T. J. Connexions for systems of parallel distributions. Quart. J. Math. Oxford Ser. (2) 7 (1956), 269–276.

In (*) same J. (1) **20** (1949), 135–145 [MR **11**, 460], A. G. Walker treated 'p-Richtungsfelder' in V^n, which are parallel to themselves. In another paper [ibid. (2) **6** (1955), 301–308; MR **19**, 312], he proved theorem 1: For any system of distributions [cf., e.g., C. Chevalley, Theory of Lie groups, Princeton Univ. Press, 1946; MR **7**, 412; p. 86] over a manifold, there exists in the large an affine connection with respect to which the distributions are parallel and which is symmetric if the system is integrable. The present paper may be regarded as a sequel to the latter paper. The author gave a short proof of the existence theorem just stated in (**) J. London Math. Soc. **32** (1957), 153–156 [MR **19**, 455]. Here he gives another proof by a method introduced in another paper [Proc. London Math. Soc. (3) **6** (1956), 191–204; MR **19**, 455], which, like Walker's, has the advantage, over the method of (**), that an explicit expression for a suitable connection is given. He then shows that the connections obtained by Walker in (*) are precisely those obtained here when all arbitrary parameters are equal to zero.

In general, there is considerable freedom in the choice of a connection which makes a system of distributions parallel; for this depends on the independent choice of an arbitrary symmetric connection and an arbitrary complementary distribution. However, a definite procedure is described which makes correspond to a given complementary distribution a connection giving parallelism whose torsion is independent of the choice of the arbitrary symmetric connection. In particular, a complete system of distributions has a system of torsion tensors canonically associated with it whose vanishing is a necessary and sufficient condition for the system to be integrable.

A system of n linearly independent vector fields (e_i) such that $[e_i, e_j]=0$ is said to form a commutative n-frame. The author proves theorem 2: A compact differentiable manifold of class C^r ($r\geqq 3$) which admits a commutative n-frame is homeomorphic to a torus. This contains, as an immediate corollary, the result proved by Ishihara and Obata [J. Math. Tokyo **1** (1953), 71–76; MR **17**, 79] that, under the author's hypothesis, the manifold has its pth Betti number equal to $\binom{n}{p}$.

T. Takasu (Yokohama)

4300:

Green, L. W. A theorem of E. Hopf. Michigan Math. J. **5** (1958), 31–34.

In this paper the following theorem is proved: Let M be a compact Riemannian manifold of class C^4 and $R=g^{ij}g^{kl}R_{ijkl}$ its scalar curvature. Suppose, moreover, that M is without conjugate points, i.e., no geodesic on M contains a pair of mutually conjugate points. Then the integral over M of R is nonpositive and it vanishes only if M is locally Euclidean (i.e., $R_{ijkl}\equiv 0$). This generalizes a theorem of E. Hopf on closed surfaces without conjugate points [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 47–51; MR **9**, 378]. The method of proof is related to that of Hopf and to earlier work of the author [Trans. Amer. Math. Soc. **76** (1954), 529–546; MR **16**, 70].

W. M. Boothby (Evanston, Ill.)

4301:

Chern, Shiing-shen; and Lashof, Richard K. On the total curvature of immersed manifolds. II. Michigan Math. J. **5** (1958), 5–12.

This paper is a sequel to one by the same authors [Amer. J. Math. **79** (1957), 306–318; MR **18**, 927]. Here the results are extended as follows. Let M^n be a compact differential manifold immersed in E^{n+N}, let $T(M^n)$ be its total curvature and $\beta(M^n)$ the sum of its Betti numbers for a given coefficient field. Then $T(M^n)\geqq\beta(M^n)$. If the equality holds in the real field, then M^n has no torsion.

Since various immersions are possible, we can consider $q(M^n)=\inf T(M^n)$. Hence $q(M^n)\geqq\beta(M^n)$. Also, let $s(M^n)$ be the minimum number of cells in a cell covering of M^n, then $s(M^n)\geqq\beta(M^n)$. For $M^2\subset E^3$, $q(M^2)=s(M^2)=\beta(M^2)$. It is conjectured that $q(M^n)=s(M^n)$.

A related problem is the characterization of immersions such that $T(M^n)=q(M^n)$. For an oriented, compact $M^2\subset E^3$ this is achieved if and only if the surface lies on one side of the tangent plane at every point of positive Gaussian curvature. Such a surface whose Gaussian curvature is $\geqq 0$ is imbedded and convex, but for $n\geqq 3$, there are non-convex, compact, orientable hypersurfaces whose Gaussian curvature is $\geqq 0$. If, however, this curvature is strictly >0, the hypersurface is imbedded and convex (Hadamard).

C. B. Allendoerfer (Seattle, Wash.)

4302:

Pogorelov, A. V. Geometric imbedding in the large of a two-dimensional Riemannian manifold into a tri-dimensional one. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 7, 156–163. (Russian. English summary)

In this paper are stated results of the author concerning generalizations of the classical Weyl problem. Proofs are only sketched; they are given in complete detail in the monograph reviewed below. The author considers a closed Riemannian manifold M of dimension two (homeomorphic to a sphere, in fact) whose Gaussian curvature K is positive and a Riemannian manifold R of dimension three. The central problems are (i) whether there exists an isometric imbedding of M in R, and (ii) to what extent the imbedding, if it exists, is unique. In the case in which R is Euclidean space this problem was posed and partially solved by H. Weyl [Vierteljahrschr. Naturf. Ges. Zürich **61** (1916), 40–72]. This problem has had a long history since that time, culminating in recent work of L. Nirenberg [Comm. Pure Appl. Math. **6** (1953), 337–394; MR **15**, 347], A. V. Pogorelov [Deformation of convex surfaces, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951; MR **14**, 400] and A. D. Alexandrov. The problem involves difficult existence theorems in partial differential equations, and the uniqueness of the solution up to rigid motions is exactly the rigidity of closed convex surfaces in Euclidean space.

To state the author's principal results, we let $K_1(x)$, $K_2(x)$ be the greatest and least sectional curvatures, respectively,

at a point x of R, and let $K^*=\max_{x\in R}\max[K_1(x), 3(K_1(x)-K_2(x))]$. Then the results are as follows. (1) If $K>K^*$, K being the (positive) Gaussian curvature of the surface M, then M can be isometrically imbedded in R. If the metrics of M and R are of class C^k, $k\geqq 6$, [resp. analytic], then the imbedding is of class $\geqq k-2$ [resp. analytic]. (2) If the image of a point together with its tangent plane and the orientation of tangent plane and normal are specified, then the imbedding is unique. In the case where R is Euclidean, no bending, i.e., no one parameter family of isometric deformations, is possible, other than rigid motions. It is shown that, roughly speaking, the same degree of uniqueness holds here, i.e., if the point and tangent plane prescribed as in (2) are moved, there is a corresponding deformation of M. In fact, given any two isometric surfaces in R, $K>K^*$, then one of them may be continually deformed into the other.

W. M. Boothby (Evanston, Ill.)

4303:

✱Погорелов, А. В. Некоторые вопросы геометрии в целом в римановом пространстве. [Pogorelov, A. V. Some questions in geometry in the large in a Riemannian space.] Izdat. Har'kovsk. Univ., Kharkov, 1957. 90 pp. 3.50 rubles.

In this monograph detailed proofs of the theorems stated in the paper reviewed above are given. Also some further results are contained in the final section. The book consists of nine sections. The first is a brief statement of the general facts about the differential geometry of surfaces, formulas for the curvature, equations of Gauss, etc. In §§ 2 and 3 a priori estimates are developed for the normal curvature of a closed surface M in a Riemannian three-dimensional manifold, which, assuming $K>K^*$ (in the notation of the preceding review) depend only on the metrics of M and R. Similar estimates are deduced for the derivatives (of all orders) of the local coordinates in R relative to the local coordinates in M. In § 4 these estimates are used to obtain the uniqueness theorem cited above. In § 5 the differential equations of the imbedding are reduced to the form of an elliptic system and in the succeeding three sections the continuity method (as given in the original paper of H. Weyl, mentioned in the preceding review) is applied to prove the existence of an isometric imbedding under the hypotheses above. This method consists of including M in a family M_t of manifolds, $M=M_1$, such that M_0 can be imbedded in R. Then it is shown that the set of values of t, $0\leqq t\leqq 1$, for which the imbedding is possible is both open and closed; hence all M_t, and in particular M_1, are imbeddable. This involves difficult existence theorems, depending on precise a priori estimates, whose deduction constitutes a major portion of the book.

The last paragraph contains generalizations of these results to cases where the regularity assumptions are very much weakened.

W. M. Boothby (Evanston, Ill.)

4304:

Kreyszig, Erwin; and Radó, Tibor. On rigidity properties of developable surfaces. J. Math. Mech. **7** (1958), 419–432.

An essential feature of the paper is a careful discussion of the basic concepts and hypotheses, which does not lend itself to reviewing.

Consider a surface C: $\mathfrak{y}(s)$ in E^3 with positive curvature; and a developable surface S: $\mathfrak{x}(s,t)=\mathfrak{y}(s)+t\mathfrak{z}(s)$, where $\mathfrak{y}''\cdot\mathfrak{z}\neq 0$. The following question is studied. Are there surfaces $\tilde{S}$: $\tilde{\mathfrak{x}}(s,t)=\mathfrak{y}(s)+t\tilde{\mathfrak{z}}(s)$, $\mathfrak{y}''\cdot\tilde{\mathfrak{z}}\neq 0$ which can be mapped on S by an intrinsically isometric mapping m which leaves C pointwise fixed? The answer is as follows: if C is a geodesic on S (and hence on $\tilde{S}$) then $\tilde{S}=S$. If the geodesic curvature of C on S (or $\tilde{S}$) does not vanish, then there is exactly one $\tilde{S}$ different from S. The $\tilde{\mathfrak{z}}(s)$ of $\tilde{S}$ is obtained from $\mathfrak{z}(s)$ of S by reflection in the osculating plane of C at $\mathfrak{y}(s)$. The surfaces S and $\tilde{S}$ are extrinsically isometric if, and only if, C is a plane curve.

H. Busemann (Cambridge, Mass.)

PROBABILITY

See also 3782, 4155, 4445.

4305:

Gnedenko, B. V. Introduction to certain problems of the theory of probability. Advancement in Math. **4** (1958), 574–582. (Chinese)

Translation of a Russian article [Ukrain. Mat. Ž. **9** (1957), 377–388; MR **19**, 776].

4306:

Kantorowitz, S.; and Kotz, S. Two-dimensional distributions with given marginals. Riveon Lematematika **11** (1957), 32–38, 91. (Hebrew. English summary)

Let F be the class of all solutions ϕ of

$$(1)\quad \iint\phi(x,y)\,dx\,dy=1,\ \int\phi(x,y)\,dy=f(x),\ \int\phi(x,y)\,dx=g(y),$$

where f and g are given non-negative functions in $L=L(-\infty,\infty)$, and all integrals run from $-\infty$ to ∞; and let F_0 be the class of "degenerate" kernels in F, i.e., the class of functions which are finite sums of products $f_i(x)g_i(y)$, with f_i, $g_i\in L$. The authors show that F is the closure of F_0 under convergence in L_1-norm, and that F is contained in the closure of F_0 under pointwise convergence almost everywhere. They also study the convergence of "marginals" f, g for convergent sequences of frequency distribution functions, and systems in which the last two equations (1) are replaced by

$$\int\phi(x,y)h(y)\,dy=f(x),\quad \int\phi(x,y)k(x)\,dx=g(y),$$

with $f, g, h, k\in L^2(-\infty,\infty)$.

A. Erdélyi (Pasadena, Calif.)

4307:

Lukacs, Eugene. On a transformation of characteristic functions. Portugal. Math. **16** (1957), 31–35.

The author shows that if $f(t)$ is the characteristic function of any probability distribution, then $-\int_0^t du\int_0^u f(y)\,dy$ is the cumulant generating function of an infinitely divisible distribution with finite variance. Various examples are given. The proof is a straightforward application of Kolmogorov's representation theorem [See, e.g., B. V. Gnedenko and A. M. Kolmogorov, Limit distributions for sums of independent random variables, Addison-Wesley, Cambridge, Mass., 1954; MR **12**, 839; **16**, 52; p. 85].

S. Katz (New York, N.Y.)

4308:

Orey, Steven. A central limit theorem for m-dependent random variables. Duke Math. J. **25** (1958), 543–546.

Central limit theorems for sums of m-dependent random variables (such that $(X_1, X_2, \cdots, X_s)$ and $(X_t, X_{t+1}, \cdots)$ are independent if $t-s>m$) have been pub-

lished under successively weaker assumptions, but even the best result so far available [P. H. Diananda, Proc. Cambridge Philos. Soc. **51** (1955), 92–95; MR **16**, 724] imposes conditions which are known to be redundant in the classical case $m=0$. The theorem proved here has conditions which, for $m=0$, reduce to the Feller-Lévy conditions (or, in the case of finite second moments, to the Lindeberg conditions), the only additional condition for $m>0$ being a restriction on the growth of the sum of the variances relative to the variance of the sum of the truncated random variables. {In relation $B1$), p. 544, delete max.}
W. Hoeffding (Chapel Hill, N.C.)

4309:

Urbanik, K. Generalized stochastic processes. Studia Math. **16** (1958), 268–334.

This paper contains detailed statements and proofs of previously announced results [Teor. Veroyatnost. i Primenen. **1** (1956), 146–149; MR **19**, 326].

L. Schmetterer (Berkeley, Calif.)

4310:

Prékopa, A. On Poisson and composed Poisson stochastic set functions. Studia Math. **16** (1957), 142–155.

Let $\mathscr{R}$ be a σ-ring of subsets of an abstract space R. For any set function $\mu(A)$ defined on $\mathscr{R}$ and any $A \in \mathscr{R}$ let $\mathrm{Var}_\mu(A)=\sup\sum_{i=1}^r|\mu(A_i)|$, where the sup is taken over every finite sequence $A_1, \cdots, A_r$ of pairwise disjoint sets $A_i \subset A\mathscr{R}$. Consider a completely additive stochastic set function $\xi(A)$ defined on $\mathscr{R}$. Suppose that the following conditions are satisfied. I. For each $A \in \mathscr{R}$ the random variable $\xi(A)$ can assume only the values of a countable set Λ of real numbers $\lambda_0=0, \lambda_1, \cdots$, such that $\lambda_i, \lambda_j \in \Lambda$ implies $\lambda_i+\lambda_j=\Lambda$. II. Define $P(\xi(A)=\lambda_k)=P_k(A)$, $k\geqq 0$. Then $\mathrm{Var}_{1-P_0}(R)<\infty$. III. $A \in \mathscr{R}$ and $1-P_0(A)>0$ imply the existence of two disjoint sets $A_i \subset A\mathscr{R}$ which fulfill $1-P_0(A_i)>0$. Using results of his paper [Acta Math. Acad. Sci. Hungar. **7** (1956), 201–213; MR **18**, 197] the author first proves that I–III imply that $\mathrm{Var}_{1-P_0}(A)$ is a bounded and atomless measure, and then the following theorem is proved: I–III imply $E(\exp(it\xi(B))=\exp(\sum_{k=1}^\infty C_k(B)(e^{i\lambda_k t}-1))$, where $C_k(B)=\int_B P_k(dA)$, and $\int_B(1-P_0(dA))$ exists also and $=\sum_{k=1}^\infty C_k(B)<\infty$. $C_k(B)$ and $\sum_{k=1}^\infty C_k(B)$ are bounded, atomless measures on $\mathscr{R}$. So $\xi(A)$ has a composed Poisson distribution. There is also another theorem of this kind. Moreover, the author studies the structure of composed Poisson stochastic set functions under somewhat more special conditions. Applications to the case $\lambda_k=k$ and to Poisson stochastic set functions are also given.

L. Schmetterer (Berkeley, Calif.)

4311:

Yaglom, A. M. Extrapolation, interpolation and filtering of stationary random processes with rational spectral density. Advancement in Math. **2** (1956), 161–201. (Chinese)

A translation of the Russian original in Trudy Moskov. Math. Obšč. **4** (1955), 333–374 [MR **17**, 167].

4312:

Yaglom, A. M. Correlation theory of processes with stationary nth increments. Advancement in Math. **2** (1956), 202–255. (Chinese)

Translation of the Russian original in Mat. Sb. N.S. **37(79)** (1955), 141–196 [MR **17**, 167].

4313:

Yaglom, A. M. Introduction to the theory of stationary random functions. Advancement in Math. **2** (1956), 3–152. (Chinese)

A translation of the Russian article in Uspehi Mat. Nauk (N.S.) **7** (1952), no. 5(51), 3–168 [MR **14**, 485].

4314:

Prékopa, A. On stochastic set functions. I. Acta Math. Acad. Sci. Hungar. **7** (1956), 215–263. (Russian summary)

English version of #4315 below.

4315:

Prékopa, András. Stochastic set functions. I. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 289–337. (Hungarian)

Let $(\Omega, \mathfrak{A}, P)$ be a probability space and $\mathbf{H}$ be an arbitrary set and denote by $\mathfrak{R}$ a ring of subsets of $\mathbf{H}$. If a random variable $\xi(\omega, A)=\xi(A)$ $(\omega \in \Omega, A \in \mathfrak{R})$ corresponds to each set $A \in \mathfrak{R}$, then $\xi(A)$ is called a random set function (r.s.f.). It is assumed that the r.s.f. has the following property: (*) If $A_1, A_2, \cdots, A_r$ are nonoverlapping sets of $\mathfrak{R}$ then the random variables $\xi(A_1)$, $\xi(A_2), \cdots, \xi(A_r)$ are independent. The r.s.f. $\xi(A)$ is said to be additive {countably additive} if $\mathrm{P}[\sum_{k=1}^r \xi(A_k)=\xi(A)]=1$ $\{\mathrm{P}[\sum_{k=1}^\infty \xi(A_k)=\xi(A)]=1\}$, where $A=\sum_{k=1}^r A_k$ $\{A=\sum_{k=1}^\infty A_k\}$ is the union of a finite {enumerable} sequence of nonoverlapping sets $A_1, \cdots, A_r$ $\{A_1, A_2, \cdots\}$.

The main purpose of the paper is to derive conditions which permit to extend a countably additive r.s.f. $\xi(A)$ defined on a ring $\mathfrak{R}$ to the smallest σ-ring $\gamma=\gamma(\mathfrak{R})$ over $\mathfrak{R}$. Conditions are given in terms of characteristic functions, of distribution functions and of quantiles of $\xi(A)$. The author uses the results of his earlier papers [Acta Math. Acad. Sci. Hungar. **7** (1956), 201–213; Publ. Math. Debrecen **4** (1956), 410–417; MR **18**, 197; **19**, 891]. The present paper concludes with a brief study of countably additive r.s.f. defined on a σ-ring. Continuous and purely discontinuous r.s.f. are defined as well as r.s.f. which are absolutely continuous with respect to a certain measure. It is shown that a countably additive r.s.f. defined on a σ-ring γ, such that the elements of $\mathbf{H}$ belong to γ, can be written as the sum of a purely discontinuous and of a continuous r.s.f.
E. Lukacs (Washington, D.C.)

4316:

Prékopa, András. Stochastic set functions. II. A new stochastic integral. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7** (1957), 339–370. (Hungarian)

The author continues his investigations of r.s.f. [see the preceding review for the notation] and defines a stochastic integral with respect to a countably additive r.s.f. $\xi(A)$ which has the property (*). The integration is taken over subsets of $\mathbf{H}$ which belong to a σ-ring γ. The integrand is a real-valued γ-measurable point function defined on $\mathbf{H}$. The integral is a generalization of the Lebesgue-Stieltjes integral and has many of its properties. However, one cannot conclude that a function $\varphi(h)$ which is majorized by an integrable function $\psi(h)$ is itself necessarily integrable. Similarly, the dominated convergence theorem is valid only under certain additional conditions. The author defines r.s.f. which are absolutely continuous with respect to an other r.s.f. and investigates some of their properties. The paper concludes with a brief discussion of expectations and variances of r.s.f. and of characteristic functionals.
E. Lukacs (Washington, D.C.)

4317:

Prékopa, A. On stochastic set functions. II, III. Acta Math. Acad. Sci. Hungar. **8** (1957), 337–374, 375–400.

Part II is essentially an English version of the author's Hungarian paper [#4316 above]. The principal addition is a brief discussion of an extension of the Radon-Nikodym theorem.

The author continues in part III his investigations of random set functions (r.s.f.) and uses the results of his earlier papers [same Acta **8** (1957), 107–126; MR **19**, 399; and #4315 above] to study the distributions of countably additive r.s.f. Let $\xi(A)=\xi(\omega, A)$ be a countably additive r.s.f.; for fixed ω this becomes a set function which is called a realization of the given r.s.f. It is then shown that a countably additive r.s.f. $\xi(A)$ which satisfies certain conditions can be written as $\xi(A)=\zeta(A)+\eta(A)$, where $\zeta(A)$ and $\eta(A)$ are both countably additive and where — with probability one — $\zeta(A)$ has purely discrete realizations while $\eta(A)$ is constant. *E. Lukacs* (Washington, D.C.)

4318:

Miehle, William. Calculation of higher transitions in a Markov process. Operations Res. **6** (1958), 693–698.

It is remarked that, for a Markov chain with a finite number of states, elementary matrix calculations can yield not only the higher transition probabilities but also the first transition probabilities and the number and nature of the paths connecting states.

D. V. Lindley (Cambridge, England)

4319:

Kendall, David G. A totally unstable denumerable Markov process. Quart. J. Math. Oxford Ser. (2) **9** (1958), 149–160.

The author uses semi-group methods to construct a Markov chain with all instantaneous states. For different methods, see D. Blackwell [Ann. Math. Statist. **29** (1958), 313–316; MR **20** #342], R. Dobrušin [Teor. Veroyatnost. i Primenen. **1** (1956), 481–485; MR **19**, 691], and W. Feller and H. P. McKean [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 351–354; MR **19**, 327].

H. P. McKean, Jr. (Cambridge, Mass.)

4320:

Jacobs, Konrad. Konjunkturschwankungen Markoffscher n-Personen-Prozesse mit monomialer Regelung. Math. Z. **69** (1958), 247–270.

Given a finite Markov chain with transition $P(j, i)=P(j, k)P(k, i)$ $(j\geqq k\geqq i)$ and supposing the existence of matrices Q such that

$$(*)\qquad P(k+1, k)=Q_1PQ_2P\cdots PQ_n,\ P=P(k, 0),\qquad (k=1, 2, \cdots),$$

the author has proved [Arch. Math. **8** (1957), 298–308; MR **19**, 988] that $P(k, 0)$ is asymptotically periodic. Here, he generalizes his result to systems of Markov chains with a view to applications to economics; the mapping (*) is replaced by a system of such maps and the intervening Q's are permitted to depend on k in an asymptotically periodic way.

H. P. McKean, Jr. (Cambridge, Mass.)

4321:

Akutowicz, Edwin J. On an explicit formula in linear least squares prediction. Math. Scand. **5** (1957), 261–266.

Let $\cdots, x_{-1}, x_0, x_1, \cdots$ be a wide sense stationary random sequence with absolutely continuous spectral function F such that $\int|\log F'(\theta)|d\theta<\infty$. Let x_n^* be the projection of x_n on the c.l.m. spanned by $x_{-1}, x_{-2}, \cdots$. Let y_{-n}, $n=1, 2, \cdots$, be the normalized projection of x_{-n} on the orthogonal manifold to $x_{-n-1}, x_{-n-2}, \cdots$. The y_n form an orthonormal sequence, and, letting $b_n=(x_0, y_{-n})$, one has $x_n^*=\sum_{\nu=n+1}^{\infty} b_\nu y_{n-\nu}$. Let $\Phi(z)=\sum_{k=0}^{\infty} b_k z^k$. Theorem 1: Under the assumption that $|F'(\theta)|$ is bounded (an assumption which, the reviewer feels, can be dispensed with), a necessary and sufficient condition that y_0 be expressible as $\sum_{k=0}^{\infty} a_k x_{-k}$ with $\sum |a_k|^2<\infty$ is that $\Phi^{-1}(z)$ can be written as $\sum_{k=0}^{\infty} c_k z^k$ with $\sum |c_k|^2<\infty$; and in this case, $a_k=c_k$, $k=0, 1, \cdots$. Theorem 2: Assuming further that either $\sum |a_k|$ or $\sum |b_k|$ is finite, one can write x_n^* as $\lim_{J\to\infty} \sum_{j=1}^{J} (\sum_{s=0}^{j} b_{n+s}a_{j-s})x_{-j}$. The reviewer was unable to follow the proof of theorem 2, but believes that the summation over s should begin at 1.

J. Feldman (Berkeley, Calif.)

4322:

Rosenblatt, M. The multidimensional prediction problem. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 989–992.

By an extension to multidimensional processes of a result by N. Wiener [Comment. Math. Helv. **29** (1955), 97–111; MR **16**, 921] it is proved that, in a weakly stationary discrete-parameter process with a multidimensional spectral density function $f(\lambda)$, the (optimum linear) one-step prediction error is positive definite whenever $\log |f(\lambda)|$ is integrable. A few immediate consequences are pointed out, and similar results are briefly stated for continuous-parameter processes.

{The author wishes to introduce the following corrections in his paper. The inequality on p. 990, l.3 from below, should read: "$|A+B|^{1/n}\geqq|A|^{1/n}+|B|^{1/n}$". Lemma 2 should read: "Let $g(\lambda)$ be an $n\times n$ nonnegative definite matrix in L. Then $|\int_{-\pi}^{\pi} g(\lambda)d\lambda|^{1/n}\geqq\int_{-\pi}^{\pi} |g(\lambda)|^{1/n}d\lambda$."

The argument following lemma 2 becomes:

$$\text{“}|E_m|^{1/n}\geqq\min_{c^{(m)}}\int_{-\pi}^{\pi} \|I-c^{(m)}(e^{-i\lambda})\|^{2/n}|f(\lambda)|^{1/n}d\lambda$$

and the expression on the right is bounded below by $2\pi \exp\{(2\pi n)^{-1}\int_{-\pi}^{\pi}\log |f(\lambda)|d\lambda\}$ by a theorem of Szegö."}

S. K. Zaremba (Swansea)

4323:

Wiener, N.; and Masani, P. The prediction theory of multivariate stochastic processes. I. The regularity condition. Acta Math. **98** (1957), 111–150.

The fundamental results of prediction theory for multivariate stochastic processes have either been stated without proof [Zasuhin, Dokl. Acad. Nauk SSSR (N.S.) **33** (1941), 435–437; MR **5**, 102] or, according to the present authors, derived with insufficient proof by Wiener [reference in #4322 above] and Doob [Stochastic processes, Wiley, New York, 1953; MR **15**, 445]. The authors first discuss in detail orthogonal series and related concepts for the Hilbert space of q-dimensional vectors with components in an L_2 space. They then apply the discussion to the treatment of a stochastic process $\{\mathbf{f}_n, -\infty<n<\infty\}$ which is stationary in the wide sense, where $\mathbf{f}_n$ is a q-dimensional vector-valued random variable. The decomposition due to Wold [A study in the analysis of stationary time series, Almqvist and Wiksell, Uppsala, 1938] in the case $q=1$ is obtained in the general case. If $\mathbf{g}_{ni}$ is the ith component of the projection of $\mathbf{f}_n$ on the closed linear manifold $\mathfrak{M}_{n-1}$ spanned by $\mathbf{f}_{n-1}, \mathbf{f}_{n-2}, \cdots$, the process is said to have as rank the rank of the matrix $\mathbf{G}=(E\{\mathbf{g}_{ni}\bar{\mathbf{g}}_{nj}\})$. The (matrix)spectral distribution function $\mathbf{F}$ is related to the Wold decomposition as in the 1-dimensional case. Generalizing the latter case [Kolmogorov, Izv. Akad. Nauk SSSR Ser. Mat. **5** (1941), 3–14;

MR 3, 4], the process has rank q if and only if log (det $\mathbf{F}'$) is summable; and has rank q and is regular ($\bigcap_{-\infty}^{\infty} \mathfrak{M}_n$ contains only the zero function) if and only if log (det $\mathbf{G}$) is equal to the integral average of log(det $\mathbf{F}'$). See also Rosenblatt [#4322 above], and Helson and Lowdenslager [#4155 above], who obtain some of the above results. {Partial errata list from authors: P. 126, 4.9 (c): interchange $\mathbf{G}(x)$ with $d\mathbf{F}(x)$. P. 128, 1.7: for $dF(\theta)$ read $F(\theta)d\theta$. P. 132, n. 3: for (4.3) read (5.3). P. 133, bottom: this argument needs modification. P. 143: in proof of 1.8(a), right continuity of $\mathbf{F}$ should be used.}

4324a:

Wiener, Norbert; et Masani, Pesi. Sur la prévision linéaire des processus stochastiques vectoriels à densité spectrale bornée. C. R. Acad. Sci. Paris **246** (1958), 1492–1495.

4324b:

Wiener, Norbert; et Masani, Pesi. Sur la prévision linéaire des processus stochastiques vectoriels à densité spectrale bornée (détermination de la fonction génératrice). C. R. Acad. Sci. Paris **246** (1958), 1655–1656.

Continuing the work of the paper reviewed above, the authors continue their generalization of the facts known for the case $q=1$, now under the restriction that the process is regular of rank q. The prediction problem is reduced to that of finding a certain matrix-valued "generating function" defined on the unit circle perimeter. If the characteristic values of the spectral distribution matrix function derivative $\mathbf{F}'$ lie in a compact subinterval of $(0, \infty)$, it is proved that the prediction of $\mathbf{f}_n$ in terms of its past is given by a single mean-convergent series. In the second paper, an iterative procedure is derived to solve the prediction problem, involving a factorization of $\mathbf{F}'$. The factorization is obtained by combining previous work by Masani [Proc. London Math. Soc. (3) **6** (1956), 59–69; MR **18**, 138] with the paper of Wiener referred to above.

J. L. Doob (Urbana, Ill.)

4325:

Wiener, N.; and Masani, P. The prediction theory of multivariate stochastic processes. II. The linear predictor. Acta Math. **99** (1958), 93–137.

Proofs of already announced results [#4324a,b above], developing the work of the authors' first paper [#4323 above]. In summary, and using the language of the latter paper, the prediction theory of a q-dimensional stationary process has now been brought to a satisfactory and usable state, at least if the process is regular and of rank q, and if the spectral density matrix function has characteristic values which lie in a compact subinterval of $(0, \infty)$. That is, the mathematical conditions for the key properties of the process have been equated with properties of the spectral distribution function, the mean square prediction error has been evaluated, and, at least under the restrictive hypotheses just stated, an algorithm for computing the prediction has been found. {Partial errata list from authors: P. 101: in eqns. (1), (2), interchange rows and columns in the square matrix; corresponding changes in the sequel. P. 114, l.5: for $\boldsymbol{\Phi}$ read $\boldsymbol{\phi}$.}

J. L. Doob (Urbana, Ill.)

4326a:

Masani, Pesi. Sur les processus vectoriels minimaux de rang maximal. C. R. Acad. Sci. Paris **246** (1958), 2215–2217.

4326b:

Masani, Pesi. Sur la prévision linéaire d'un processus vectoriel à densité spectrale non bornée. C. R. Acad. Sci. Paris **246** (1958), 2337–2339.

The author continues the joint work with Wiener [#4323, 4324a,b above] on prediction theory for multivariate stochastic processes. The terminology of the previous reviews will be used. Proofs are sketched of the following theorems. If $\{\mathbf{f}_n, -\infty<n<\infty\}$ is a stationary vector process, it is called minimal of maximal rank if the projection of $\mathbf{f}_n$ on the closed linear manifold spanned by the remaining random variables is not $\mathbf{f}_n$, and if, when $\boldsymbol{\phi}_n$ is $\mathbf{f}_n$ less this projection, the matrix $(\boldsymbol{\phi}_n, \boldsymbol{\phi}_n)$ is nonsingular. In this case the inverse of the matrix generating function belongs to L_2^{0+}. A process is minimal of maximal rank if and only if the spectral density matrix $\mathbf{F}'$ is nonsingular almost everywhere and if $\mathbf{F}'^{-1} \in L_1$. This (interpolation) theorem is due to Kolmogorov [Byull. Moskov. Gosudarstv. Univ. Mat. **2** (1941), no. 6; MR **5**, 101] in the one-dimensional case. Under the hypothesis that $\mathbf{F}'^{-1} \in L_1$ and that the ratio of maximum to minimum characteristic value of $\mathbf{F}'$ is in L_1, an algorithm is given for finding the generating function of the process. Finally, if $\mathbf{F}' \in L_\infty$ and if $\mathbf{F}'^{-1} \in L_1$, a representation of the form $\sum E_{\nu k}\mathbf{f}_{-k}$ is found for the projection of $\mathbf{f}_\nu$ (prediction with lag ν) on the closed linear manifold spanned by $\{\mathbf{f}_k, k\leq 0\}$. This result strengthens one in the first reference above. {Erratum: P. 2338, l.2 bottom: for $\boldsymbol{\Phi}^{-1}$ read $\tilde{\boldsymbol{\Phi}}^{-1}$.}

J. L. Doob (Urbana, Ill.)

4327:

Chiang, Tse-Pei. Extrapolation theory of a homogeneous random field with continuous parameters. Teor. Veroyatnost. i Primenen. **2** (1957), 60–91. (Russian. English summary)

The author discusses prediction theory for a stochastic process $\{x(s, t), -\infty<s, t<\infty\}$, where s, t are real numbers, there is second-order stationarity, and a mean square continuity condition is satisfied, so that the covariance function is given by

$$M\{x(s+m,t+n)\bar{x}(m,n)\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(s\lambda+t\mu)}dF_x(\lambda, \mu).$$

The linear least squares prediction problem, as formulated by the author, is that of approximating the vector $x(s, t)$ for $t>0$ by its projection on the closed linear manifold of random variables spanned by $x(u, v)$, with $v\leq 0$ but no restriction on u. If this projection goes to 0 when $t\to\infty$, the process is called regular; if the projection is $x(s, t)$, the process is called singular. The analogues of the known results for the one-dimensional parameter case are derived. For example, if g is the derivative of the absolutely continuous component of $dF_x(\lambda, \mu)$ measure with respect to $dF_x(\lambda, \infty)d\mu$ measure, the process is singular if and only if $\int_{-\infty}^{\infty} g(\lambda, \mu)/(1+\mu^2)\, d\mu=\infty$ for almost all λ [$dF_x(\lambda, \infty)$ measure]. If the process is regular, the above two-dimensional measures are absolutely continuous with respect to each other, and the above integral is finite for almost all λ ($dF_x(\lambda, \infty)$ measure). An interesting special case, in which the results assume a particularly perspicuous form is that of processes "of Markov type" in which the projection described above is in the closed linear manifold of random variables spanned by $x(u, 0)$, $-\infty<u<\infty$. In an earlier paper [Dokl. Akad. Nauk SSSR **112** (1957), 207–210; MR **19**, 327] the author discussed the corresponding problems for

processes with parameter ranging on the points of a plane with integral coordinates. A prediction problem of a slightly different character for this type of process has also been treated by Helson and Lowdenslager [#4155 above]. *J. L. Doob* (Urbana, Ill.)

4328:

Wiener, N.; and Akutowicz, E. J. The definition and ergodic properties of the stochastic adjoint of a unitary transformation. Rend. Circ. Mat. Palermo (2) 6 (1957), 205–217; addendum, 349.

Let $X(t, \alpha)$ be the complex separable Brownian motion process having the characteristic properties: $X(0, \alpha)=0$; $E\{X(t, \alpha)\}=0$, $E\{|X(t, \alpha)-X(s, \alpha)|^2\}=|s-t|$, $-\infty<s$, $t<\infty$; $E\{(X(t_2, \alpha)-X(t_1, \alpha))\overline{(X(s_2, \alpha)-X(s_1, \alpha))}\}=0$, $t_1\leqq t_2\leqq s_1\leqq s_2$. The authors prove that, given any unitary transformation U in $L^2(-\infty, \infty)$, there is an essentially unique measure-preserving point transformation T on the α's, which is 1-1 and satisfies $\int_{-\infty}^{\infty} U\varphi(t)dX(t, \alpha)=\int_{-\infty}^{\infty} \varphi(t)dX(t, T_\alpha)$ for any $\varphi \in L^2$, excepting null sets. T is called the stochastic adjoint of U. 1. If U^λ, $-\infty<\lambda<\infty$, is a group of unitary transformations, then the point spectrum of U^λ is absent if and only if the associated group T^λ is weakly mixing. 2. If the spectrum of U^λ is absolutely continuous, then T^λ is strongly mixing. 3. The converse of 2 is false. The proofs are based on the following theorem: A stationary (wide sense) Gaussian process $x(t, w)$, $w \in \Omega$, $-\infty<t<\infty$, is weakly mixing [strongly mixing] if and only if for arbitrary $t_1, \cdots, t_N$, the elements $\rho_{ij}(\lambda)=\int_\Omega x(t_i+\lambda, w)\overline{x(t_j, w)}dw$, of the $2N$-dimensional covariance matrix $\mathrm{P}(\lambda)$ of $x(t_1, w), \cdots, x(t_N, w), x(t_1+\lambda, w), \cdots, x(t_N+\lambda, w)$, satisfy $\lim c^{-1}\int_{-c}^{c} |\rho_{ij}(\lambda)|^2 d\lambda=0$, $c\to\infty$ [$\lim \rho_{ij}(\lambda)=0$, $\lambda\to\infty$]. *M. Cotlar* (Buenos Aires)

4329:

Feller, William. The numbers of zeros and of changes of sign in a symmetric random walk. Enseignement Math. (2) 3 (1957), 229–235.

Let $S_n=X_1+X_2+\cdots+X_n$ where the X_j's are independent and assume the values ±1 with probability $\frac{1}{2}$. The author derives for this symmetric random walk explicit formulas for the probability distribution of the number of returns to the origin, the number of changes of sign and other related quantities. The derivations are of a very elementary nature and the paper is self-contained. A more exhaustive treatment appears in Chapter III of the second edition of the author's book [W. Feller, An introduction to probability theory and its applications, second edition, Wiley, New York, 1957; MR 19, 466]. *J. L. Snell* (Palo Alto, Calif.)

4330:

Theodorescu, Radu. Sur certains processus à liaisons complètes. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 24 (1958), 260–262.

The author considers continuous parameter processes with stationary transition probabilities. Let $P_t(\omega, X^{[0,u]})$ be (roughly) the probability that, for a sample function ω given to time 0, the restriction of the sample function to $[t, t+u]$ is the translation by t of a member of a class $X^{[0,u]}$ of functions on $[0, u]$. It is stated that, if (a) there is a function ε, positive and summable on $[0, \infty)$, such that, for sufficiently large s, $\varepsilon(s)<1$ and

$$|P_t(\omega_1, X^{[0,u]})-P_t(\omega_2, X^{[0,u]})|\leqq\int_s^\infty \varepsilon(\alpha)d\alpha,$$

whenever $t\geqq0$ and ω_1 and ω_2 are identical sample functions on the interval $[-s, 0]$; and if (b) for all $\omega, \omega', X^{[0,u]}$, $P_0(\omega, X^{[0,u]})/P_0(\omega', X^{[0,u]})\geqq\lambda>0$; then $\lim_{t\to\infty} P_t(\omega, X^{[0,u]})$ exists and is independent of ω. If the transition probabilities are determined by densities, the corresponding result holds for densities. If the process is a Markov process, (a) is automatically satisfied; the theorem was proved in this case by Kolmogorov [Math. Ann. 104 (1931), 415–458]. *J. L. Doob* (Urbana, Ill.)

4331:

Motoo, Minoru. Proof of the law of iterated logarithm through diffusion equation. Ann. Inst. Statist. Math. 10 (1958), 21–28.

"The iterated logarithm type theorems for the absolute value of k-dimensional Brownian motion ($k=1, 2, \cdots$) have been given by many authors in various ways. The aim of this paper is to prove these theorems by a unified method.

Let $X(t)$ be a diffusion process in an interval, and let $\bar{C}_1$, $\bar{C}_2$, ($\underline{C}_1$, $\underline{C}_2$) be the classes of increasing (decreasing) functions in $(0, \infty)$ such that

$$\bar{C}_1=\{\varphi:P\{\limsup_{t\to\infty} (X(t)-\varphi(t))\geqq0\}=1\},$$

$$\bar{C}_2=\{\varphi:P\{\limsup_{t\to\infty} (X(t)-\varphi(t))\geqq0\}=0\},$$

$$\underline{C}_1=\{\psi:P\{\liminf_{t\to\infty} (X(t)-\psi(t))\leqq0\}=1\},$$

$$\underline{C}_2=\{\psi:P\{\liminf_{t\to\infty} (X(t)-\psi(t))\leqq0\}=0\}.$$

In section 2 we completely determine the classes $\bar{C}_1$, $\bar{C}_2$, $\underline{C}_1$ and $\underline{C}_2$, if $X(t)$ is recurrent and the expectation of its recurrence time is finite. For example, Uhlenbeck's process satisfies this condition.

In section 3 we treat the absolute value of k-dimensional Brownian motion. This process is a one-dimensional diffusion process which does not satisfy the above conditions, but, by change of scale and time, we can apply the theorem of section 2 and prove the iterated logarithm type theorems." (Author's summary)

J. Wolfowitz (Ithaca, N.Y.)

4332:

Sevast'yanov, B. A. Branching stochastic processes for particles diffusing in a bounded domain with absorbing boundaries. Teor. Veroyatnost. i Primenen. 3 (1958), 121–136. (Russian. English summary)

Particles diffuse independently, according to the Brownian motion without drift, in a bounded r-dimensional region G with absorbing boundary Γ. A particle has a probability $cP_n dt$ of being transformed during the time interval $(t, t+dt)$ into n particles, $n=0, 2, 3, \cdots$, $P_0+P_2+P_3+\cdots=1$, and a probability $1-cdt$ of not being transformed. Put $P_1=0$, $F_1(z)=\sum_{n=0}^{\infty} P_n z^n$, (author's notation changed slightly), and let $P_n(x, t)$ be the probability that the posterity of a particle at x, after t generations, consists of n particles. Let $K(x, y)dy$, $x, y \in G$, be the probability that a particle initially at x is ultimately transformed in the volume element dy at y. Let $h(x)=1-\int_G K(x, y)dy$ and $F(x, t, z)=\sum_{n=0}^{\infty} P_n(x, t)z^n$. Then the following basic equation is satisfied:

$$F(x, t+1, z)=\int_G K(x, y)F_1[F(y, t, z)]dy+h(x).$$

Let $m(x, y, t)$ be the expected density of particles born at y in the tth generation; then m satisfies the diffusion equation $\partial m/\partial t=D\Delta m+am$, where $a=c[F_1'(1)-1]$, Δ is the Laplacian, and D is the diffusion constant. The solution is given in terms of the eigenfunctions of (*) $\Delta v+$

$\lambda v=0$; $v(y)=0$, $y\in\Gamma$. If $t\to\infty$, then $F(x,t,0)\to P_0(x)$, the probability of extinction, which satisfies (**) $P_0(x)=\int_G K(x,y)F_1[P_0(y)]dy+h(x)$. The process is called degenerate if $P_0\equiv1$. Theorem 1. If (**) has a solution $u(x)$, $0\le u\le1$, $u\ne1$, then the process is not degenerate, and $P_0\le u$. Theorem 4. The process is degenerate if and only if $c[F_1'(1)-1]\le D\lambda_1$, where λ_1 is the first eigenvalue of (*). Similar results can be obtained for processes with several types of particles. A detailed study is made of the case where there are two types T and T_0, where T_0 is a "final" type that undergoes no further transformation and is not absorbed on the boundary.

T. E. Harris (Santa Monica, Calif.)

4333:

★Freeman, J. J. Principles of noise. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London; 1958. x+299 pp. $9.25.

A textbook for a graduate course in electrical engineering. Chapter headings are: Fourier series and integrals, probability, stationary random processes, physical sources of noise, equivalent noise generators, noise factor, the measurement of a direct voltage, the Gaussian random process, the detection of alternating waveforms, target noise. Topics treated in detail include Campbell's theorem, Nyquist's law, noise in two- and four-terminal networks and in vacuum tube amplifiers. A great number of examples are worked out in the text as part of the exposition, and problems are given at the end of most sections.

S. P. Lloyd (Murray Hill, N.J.)

STATISTICS

See also 3912.

4334:

Dhruvarajan, P. S.; and Singal, M. K. A note on moments and cumulants. Math. Student 25 (1957), 27–32.

Using a generalization of the expansion of e^x by Mahajani, the authors have obtained expressions for moments about the origin in terms of cumulants in the form of a determinant that is easy to work with. Using a similar generalization of the expansion of $\log(1+x)$, also by Mahajani, they have obtained a known expression in the form of a determinant for cumulants in terms of moments.

O. P. Aggarwal (Santiago)

4335:

Liserre, Guido O. G. A study of the g and d statistics. Rev. Un. Mat. Argentina **17** (1955), 91–95 (1956). (Spanish)

4336:

Kulldorff, Gunnar. On the conditions for consistency and asymptotic efficiency of maximum likelihood estimates. Skand. Aktuarietidskr. **1957**, 129–144 (1958).

Let X be a random variable with density function $f(x,\theta)$, $\theta\in\Omega$, an open interval of the real line. If $x_1, x_2, \cdots, x_n$ are independent observations of X, the equation $\partial\log L/\partial\theta=0$, where $L=\prod_{i=1}^n f(x_i,\theta)$, is called the maximum likelihood equation (m.l.e.).

Denote by (*) the proposition that for sufficiently large n the m.l.e. has a unique solution, $\hat\theta$, which is consistent and such that $n^{\frac12}(\hat\theta-\theta)$ is asymptotically normal with mean zero and variance

$$\left(\int_{-\infty}^{\infty}\left[\frac{\partial\log f(x,\theta)}{\partial\theta}\right]^2 f(x,\theta)dx\right)^{-1}.$$

Numerous proofs have been given of all or part of (*), under a variety of assumptions; also most of these proofs contain holes of assorted sizes.

In this paper two new proofs of (*) are given under new sets of conditions; in one case it is not necessary to assume the existence of $\partial^3\log f(x,\theta)/\partial\theta^3$ as, for example, was the case in Cramér's proof [Mathematical methods of statistics, Princeton Univ. Press, Princeton, N.J., 1946; MR 8, 39; pp. 500–504]. The difficulty that Wald [Ann. Math. Statist. **20** (1949), 595–601; MR 11, 261] pointed out in Cramér's proof, viz., possible multiplicity of solutions of the m.l.e., is avoided by postulating at most one solution. The author appears to be unaware of the paper of LeCam [Univ. California Publ. Statist. **1** (1953), 277–329; MR 14, 998], where a statement of the same theorem is given under another set of conditions not involving $\partial^3\log f(x,\theta)/\partial\theta^3$. In view of other results in the paper of LeCam just cited, the present author's use of the term "asymptotically efficient in the strict sense" for property (*) is questionable. *D. G. Chapman* (Seattle, Wash.)

4337:

Breitenberger, E. Remarks on the least-squares reduction of angular correlation data. Proc. Phys. Soc. Sect. A. **69** (1956), 489–491.

Mathematically the problem considered is that of estimating parameters A, c_2 and c_4 from data consisting of independent random variables k_i ($i=1,\cdots,m$), with $E(k_i)=A[1+c_2P_2(\cos\delta_i)+c_4P_4(\cos\delta_i)]$ and known variances, made at angles δ_i. The author suggests that the usual least-squares estimates obtained by estimating A, Ac_2, Ac_4, respectively, as b_0, b_2, b_4 (by the usual linear method), and hence c_2 and c_4 as b_2/b_0 and b_4/b_0, are not satisfactory. He proposes instead an iterative method. The argument appears to be open to criticism. The estimates b_2/b_0 and b_4/b_0 are maximum likelihood estimates and their approximate standard errors can be calculated from the correct form of the author's equation (4),

$$\mathrm{var}(b_2/b_0)= b_0^{-2}\,\mathrm{var}(b_2)-2b_2b_0^{-1}\,\mathrm{cov}(b_2,b_0)+b_2^2b_0^{-4}\mathrm{var}(b_0).$$

D. V. Lindley (Cambridge, England)

4338:

Yoneda, Keizo; and Uchiyama, Moritune. Some estimations in the case of relatively large class intervals. Yokohama Math. J. **4** (1956), 99–118.

Two methods are presented for estimating the mean and standard deviation of a normal distribution when the observations are coarsely grouped. One is the maximum-likelihood method based only on the number of observations in each group. The computational technique proposed is essentially Newton's iterative method. Accompanying tables reduce the work required. Another method is appropriate for the case of three intervals. The observed relative frequencies for each cell are equated with the probability, yielding two equations from which the estimates are easily derived. If there are more than three cells, one may join groups together until only three cells are left. *H. Chernoff* (Stanford, Calif.)

4339a:

Doornbos, R.; and Prins, H. J. On slippage tests. I. A general type of slippage test and a slippage test for normal variates. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 38–46.

4339b:

Doornbos, R.; and Prins, H. J. On slippage tests. II. Slippage tests for discrete variates. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 47–55.

Theorem (p. 48). Suppose the integer valued random variables, $U_1, \cdots, U_k$, are independently distributed. If

$$\frac{P[\sum_{i=1}^{k} U_i-U_m-U_n=a]}{P[\sum_{i=1}^{k} U_i-U_m-U_n=a+1]} \qquad (m\neq n),$$

where a is an integer, is a non-decreasing function of a, then

$$P\Big[U_m\geqq u_m \text{ and } U_n\geqq u_n\Big|\sum_{i=1}^{k} U_i=N\Big]\leqq$$
$$P\Big[U_m\geqq u_m\Big|\sum_{i=1}^{k} U_i=N\Big]\times P\Big[U_n\geqq u_n\Big|\sum_{i=1}^{k} U_i=N\Big],$$

for every pair of integers u_m and u_n and for every non-negative integer N.

This and similar results are used in Bonferroni's inequality for approximating probabilities associated with slippage tests. *I. R. Savage* (Minneapolis, Minn.)

4340:

Ford, L. R., Jr. Solution of a ranking problem from binary comparisons. Amer. Math. Monthly **64** (1957), no. 8, part II, 28–33.

Ford has acknowledged [same Monthly **65** (1958), 514] the similarity of this paper to that of Bradley and Terry [Biometrika **39** (1952), 324–345; MR **17**, 56–57]. Also see Bradley [ibid. **41** (1954), 502–537; **42** (1955), 450–470; MR **17**, 57, 1103]. *I. R. Savage* (Minneapolis, Minn.)

4341:

Dalenius, Tore; and Hodges, Joseph L., Jr. The choice of stratification points. Skand. Aktuarietidskr. **1957**, 198–203 (1958).

A stratification of a univariate population with finite variance, distributed over the interval (a, b) according to the density f, is defined by the stratification points x_h, where $a<x_0<x_1<\cdots<x_{L-1}<x_L=b$, with

$$W_h=\int_{x_{h-1}}^{x_h} f(x)dx,\quad W_h\mu_h=\int_{x_{h-1}}^{x_h} xf(x)dx,$$
$$W_h(\sigma_h^2+\mu_h^2)=\int_{x_{h-1}}^{x_h} x^2f(x)dx.$$

Dalenius and Gurney [same Aktuarietidskr. **34** (1951), 133–148; MR **14**, 64] conjectured that $W_h\sigma_h$=constant would provide an approximately optimum stratification. The present paper proves a theorem which supports this conjecture in some sense.

Considering the new variable y defined by $y=G(x)=\int_a^x (f(u))^{\frac{1}{2}}du$, $K=G(b)<\infty$, and introducing the stratification points x_h^* defined by $G(x_h^*)=hK/L$, the authors show that $S^*/S_0\to 1$ and $LS_0\to K^2/\sqrt{12}$ as $L\to\infty$, where the corresponding values of σ, S_h, are denoted by asterisks for the stratification corresponding to $\{x_h^*\}$, while S_0 denotes the minimum value of $S=\sum_{h=1}^{L} W_h\sigma_h$ among all possible stratifications for an assigned L.

T. Kitagawa (Fukuoka)

NUMERICAL METHODS

See also 4055, 4073, 4199, 4207, 4358, 4380.

4342:

Ford, L. R., Jr.; and Fulkerson, D. R. A suggested computation for maximal multi-commodity network flows. Management Sci. **5** (1958), 97–101.

"A simplex computation for an arc-chain formulation of the maximal multi-commodity network flow problem is proposed. Since the number of variables in this formulation is too large to be dealt with explicitly, the computation treats non-basic variables implicitly by replacing the usual method of determining a vector to enter the basis with several applications of a combinatorial algorithm for finding a shortest chain joining a pair of points in a network." (Author's summary)

M. M. Flood (Ann Arbor, Mich.)

4343:

Dantzig, George; Johnson, Selmer; and White, Wayne. A linear programming approach to the chemical equilibrium problem. Management Sci. **5** (1958), 38–43.

"The well-known chemical equilibrium problem is expressed in the form of minimizing the free energy of a mixture in order to compute the chemical composition at equilibrium. By piece-wise linear approximations to the free energy function, the problem becomes a linear program which can be solved by a standard code on a computing machine. Successive approximations give any degree of accuracy." (Author's summary)

M. M. Flood (Ann Arbor, Mich.)

4344:

Burdina, V. I. On a method of solving simultaneous linear algebraic equations. Dokl. Akad. Nauk SSSR **120** (1958), 235–238. (Russian)

A modification of the Hestenes-Stiefel conjugate directions method for the solution of alinear system $Ax=k$ is proposed and a process is described by which its inherent errors can be reduced in steps. Let $a^{(i)}$ $(i=1, \cdots, n)$ be the columns of the real $n\times n$ matrix A and $|A|\neq 0$. They are replaced by the following system of orthogonal vectors

$$b^{(i)}=a^{(i)}+\sum_{s=1}^{i-1} f_s^{(i)}a^{(s)},\quad f^{(i)}=e^{(i)}-\sum_{s=1}^{i-1}\gamma_s^{(i)}f^{(s)},$$
$$\gamma_s^{(i)}=\frac{a^{(i)\prime}b^{(s)}}{b^{(s)\prime}b^{(s)}}$$

$(s=1, 2, \cdots, i-1;\ i=1, 2, \cdots, n)$, where the $e^{(i)}$ are the coordinate unit vectors. Now the vector $k=a^{(n+1)}$ is adjoined to the system $a^{(1)}, \cdots, a^{(n)}$ and the orthogonalization is applied once more; it turns $a^{(n+1)}$ into $0=k+\sum_{i=1}^{n} f_i^{(n+1)}a^{(i)}$, where $f^{(n+1)}=-\sum_{i=1}^{n}\alpha_i f^{(i)}$ and $\alpha_i=k'b^{(i)}/b^{(i)\prime}b^{(i)}$. Therefore, $x=-f^{(n+1)}$ represents the solution of the system $Ax=k$. It is now observed that in numerical applications these formulae can lead only to an approximate solution because of the inherent rounding-off errors. Let ${}^{\mathrm{I}}b^{(i)}$ be the first approximation to $b^{(i)}$ and $\Delta_{ij}={}^{\mathrm{I}}b^{(i)\prime}\,{}^{\mathrm{I}}b^{(j)}$. In the case of an ill-conditioned matrix A these will deviate considerably from zero if $i\neq j$. By applying orthogonalization again to the ${}^{\mathrm{I}}b^{(i)}$, obtain the

vectors ${}^{II}b^{(i)}$, etc. It is shown that if $|\Delta_{pq}/\Delta_{pp}|\leq n^{-1}$, $p<i$, $q<i$, $p\neq q$, then the sequence ${}^{I}b^{(i)}$, ${}^{II}b^{(i)}$, ${}^{III}b^{(i)}$, $\cdots$ has the true vector $b^{(i)}$ as its limit. A second theorem (without proof) states an estimate of the errors $|\bar{x}_i-x_i|$ assuming that $|\Delta_{ij}/\Delta_{ii}|<\frac{1}{2}n^{-1}$.

H. Schwerdtfeger (Montreal, P. Q.)

4345:

Raichl, Jiří. The economical coding of high-order matrices for automatic computers. Stroje na zpracování informací 4 (1956), 257–271. (Czech. Russian and English summaries)

"The work discusses methods of economical coding of matrices characterised by some special property. The case of a matrix in which the majority of the elements is zero, is discussed in detail. The characteristic, i.e., a matrix of the same number of rows and columns in which elements corresponding to zero elements are likewise zero, and elements corresponding to non-zero elements are equal to unity, is formed. Individual rows of this characteristic are placed in the memory in the form of binary numbers. Only the non-zero elements of the original matrix are then placed in the memory. The second example is that of the matrix of a finite-difference analog of the Laplace equation in a region of irregular shape. The matrix is here characterised by a difference operator and a set of numbers expressing the number of points by which a given line exceeds the preceding line both on the right and the left." (Author's summary)

H. Schwerdtferger (Montreal, P. Q.)

4346:

Lyaško, A. D. The convergence of Galerkin type methods. Dokl. Akad. Nauk SSSR **120** (1958), 242–244. (Russian)

4347:

Derwidué, L. Une méthode par séparation de calcul des racines complexes des équations algébriques. Mathesis **66** (1957), 354–359.

In a previous paper [Mathesis **66** (1957), 144–151; MR **21** #125] the author develops an algorithm of H. Bückner [Quart. Appl. Math. **10** (1952), 205–213; MR **14**, 145] for what amounts to a calculation of the Routh-Hurwitz determinants. By successively reducing the roots of the equation and applying the algorithm, it is possible to separate the roots into strips parallel to the imaginary axis; hence, to approximate the real parts of the roots. Knowing the real parts of any complex conjugate pair, it is a simple matter to get the imaginary parts. Initial approximations thus obtained can be improved by Newton's method. Examples seem to indicate that a separation sufficient for a start with Newton's method is possible with only relatively low accuracy for the real parts.

A. S. Householder (Oak Ridge, Tenn.)

4348:

Gol'cov, N. A. The use of a certain functional series in deducing formulas involved in various numerical methods of solving ordinary differential equations. Dokl. Akad. Nauk SSSR **120** (1958), 450–453. (Russian)

Tabulation of coefficients and error formulas for applying to differential equations certain interpolation formulas of the form

$$y(x_n+h)=\sum_k \sum_i h^k C_{ki} y^{(k)}(x_n+\alpha_{ki}h).$$

A. S. Householder (Oak Ridge, Tenn.)

4349:

Patzelt, Gerhard. Über die Gewinnung einer gewissen Klasse von Partikulärlösungen bei bestimmten Typen gewöhnlicher oder partieller Differentialgleichungen beliebiger Ordnung. Z. Angew. Math. Mech. **36** (1956), 257–258.

4350:

Hahn, Susan G. Stability criteria for difference schemes. Comm. Pure Appl. Math. **11** (1958), 243–255.

Es werden verschiedene Stabilitätskriterien für Differenzenverfahren bei symmetrischen hyperbolischen Systemen von linearen partiellen Differentialgleichungen erster Ordnung miteinander verglichen. Das Differentialgleichungssystem sei $u_t=\sum_{k=1}^{l} A^k u_{x^k}$, die Differenzengleichungen lauten $u(x,t+\Delta t)=\sum_{j=1}^{m} C^j(x+\Delta t r_j, t)$ mit $\sum C^j=I$, $\sum C^j r_j{}^k=A^k$. Dabei bedeutet u einen Vektor mit p Komponenten und x den Vektor $x=(x^1, \cdots, x^l)$, die A^k und C^j sind $p\times p$ Matrizen und die r_j beliebige Vektoren mit l Komponenten. Das Hauptergebnis ist, dass die von von Neumann und von Courant-Friedrichs-Levi angegebenen notwendigen Bedingungen für die Stabilität und die hinreichende Bedingung von Friedrichs äquivalent sind, falls die Zahl $l+1$ der unabhängigen Variablen gleich der Anzahl m der Vektoren r_j ist und diese Vektoren nicht in einer $(l-1)$-dimensionalen Hyperebene liegen. [Eine Uebersicht über Stabilitätskriterien findet man bei Lax–Richtmyer, dieselben Comm. **9** (1956), 267–293; MR **18**, 48.] Für den Fall $m>l+1$ wird am Beispiel der Wellengleichung gezeigt, dass die Bedingung von Friedrichs und die von Courant-Friedrichs Levi nicht gleichzeitig erfüllt zu sein brauchen. *J. Schröder* (Hamburg)

4351:

Aleksidze, M. A. On the rate of convergence of the iteration process in the case of a difference solution of the Dirichlet problem for Laplace's equation. Dokl. Akad. Nauk SSSR **120** (1958), 9–12. (Russian)

Relative computing times are estimated, taking account of the number of long operations (divisions) and of short operations (addition), and using over-relaxation with the single-step and the total-step iteration.

A. S. Householder (Oak Ridge, Tenn.)

4352:

Aleksidze, M. A. On the expediency of the use of the alternating Schwarz method in digital electronic computers. Dokl. Akad. Nauk SSSR **120** (1958), 231–234. (Russian)

Continuing the paper reviewed above, detailed estimates are made of the time required for solving the Dirichlet problem by use of certain iterative schemes. The method of Schwarz utilizes the solution for two subregions to obtain the solution for their join [Sobolev, Dokl. Akad. Nauk SSSR 13 (1936), 243–246].

A. S. Householder (Oak Ridge, Tenn.)

4353:

Lyusternik, L. A. A finite-difference analog of Green's function in the three-dimensional case. Vyčisl. Mat. **1** (1957), 3–22. (Russian)

The author considers a class of finite difference operators which includes finite difference analogues of the Laplacian. For the finite-difference analogue of the Dirichlet problem in three dimensions, the author constructs the analogue of the Green's function, and proves that, as the finite grid is refined so that the grid-size goes to zero, the finite-difference Green's function approaches the classical continuous Green's function, uniformly if we exclude a neighborhood of the singular point. The proof involves many intricate estimations,

and is facilitated somewhat by the choice of an appropriate basis for the space of functions defined on the grid points. *R. B. Davis* (Syracuse, N.Y.)

4354:

Hovanskiĭ, G. S. Forms of dependence having added possibilities for transformation of charts with oriented transparent sheet. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 251–254. (Russian)

4355:

Nikolaev, P. V. On the uniqueness of nomographic representations of equations. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 31–34. (Russian)

4356:

Smorkačev, E. T. Some kinds of local nomograms. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 880–883. (Russian)

COMPUTING MACHINES

See also 3774, 3775, 4345.

4357:

***Raymond, F.-H. L'automatique des informations. Principes des machines (à calculer, en particulier) opérant sur de l'information.** Collection Évolution des Sciences, 9. Masson et Cie, Éditeurs, Paris, 1957. xiii+187 pp. 1600 fr.

This book is probably the most novel recent analysis of computers, both analog and digital, that the reviewer has seen. It is of note that a French engineer-mathematician, working outside the main stream of digital computer research that up until recently has been concentrated in the U.S.A., Great Britain, and the Soviet Union, has had time to sit down and evaluate and attempt to formalize information machines, particularly digital ones, in a semi-mathematical fashion.

The book is one of the series "Evolution of the sciences", meant specifically for the French-reading scientific public in general; nevertheless, it has a breadth that recommends it to both workers in the field and mathematicians who would like to know something for the first time about machines.

The book does not go too deeply into the subject of any of its chapter headings: An introductory panorama on automaticity; Numerical representations; Structure of analog computers; Coding and representations of information; Fine structure of a digital machine; What is an automatic calculator?; A basic digital structure — from the concrete to the abstract; and finally, Programming. In none of these are the latest ideas in these fields considered. Nevertheless, the leisured approach to some computer fundamentals proposes new ideas, because of the abstract approach of which more practical machine users and designers may not have thought. Particularly stimulating are the formal description of a digital computer and an operations research analysis of the gain in using "automatic programming" with digital machine solutions.

Unfortunately, the book has many typographical and even mathematical errors. Despite this, it presents a fresh approach in an area of investigation where the pace of technological progress has not allowed much time to think about what is being accomplished.

J. W. Carr, III (Chapel Hill, N.C.)

4358:

Valach, Miroslav. The translation of numbers from the system of remainder classes to a polyadic system by change of scale of period. Stroje na zpracování informací **4** (1956), 53–64. (Czech. Russian and English summaries)

This work contains the theoretical bases for the translation of numbers from the remainder class system to a polyadic system by means of a so-called change of scale of period. It is the basis for establishing rapid contact networks required in machines computing in the remainder class system. *From the author's summary*

4359:

Kantorovich, L. V. On a system of mathematical symbols, convenient for electronic computer operations. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 738–741. (Russian)

This is a discursive essay on general principles. The author's main idea is that any mathematical calculation — even in a generalized non-numerical sense — deals with a finite number of objects; that these, regardless of their actual nature, can be designated by numbers; that the connections between these can be exhibited in the form of a replacement scheme allowing certain replacements of a number by a finite sequence. He does not go into any detail as to how this idea is to be applied, but merely mentions how a variety of problems, including problems of automatic programming, can be looked at from that point of view. *H. B. Curry* (University Park, Pa.)

4360:

Kantorovič, L. V. On carrying out numerical and analytic calculations on machines with programmed control. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **10** (1957), no. 2, 3–16. (Russian. Armenian summary)

The author describes a certain system of programming for digital computers developed at the Leningrad Division of the Mathematical Institute of the Academy of Sciences of the USSR, and then discusses the "systematic carrying out of certain analytic computations with the aid of universal computing machines". This paper is an expansion of a short report given previously [#4359 above] and contains a more detailed but not complete discussion of the programming system ("prorab") first mentioned there.

The system of programming allows problems to be expressed in a system of unary or binary operators, thus:

$$\begin{aligned} 1&=(+,2,3)\\ 2&=(\times,4,5)\\ 4&=(\surd,6)\\ 3&=(\#/,2,4)\\ 6&=z\\ 5&=a \end{aligned}$$

stands for

$$y=((\surd z)\times a)+[((\surd z)\times a)/\surd z].$$

These sequences, which constitute descriptions of algorithms, may be transformed by minimization processes, as well as the basic laws governing the operators (associativity, distributivity), into equivalent sequences. They are apparently performed in an "interpretive" fashion, as compared with the "programming-program" (compiler-translator) developed at Moscow by the Computing Center. The elements designated by the numbers

may be either real numbers, more elaborate mathematical objects such as complex numbers, vectors, or matrices, or operators themselves. The author describes the form of introduction into the machine of such elements, and produces as an example apparently solved on a Moscow computer (BESM I?): numerical solution of an integral equation with non-symmetric kernel. The "prorab" described is one of five completed, using 500–600 memory cells; others are termed "more practical".

The final part of the discussion is on a working program for bringing rational functions into canonical form, and performing differentiation on such functions. He proposes extending the techniques to solution of differential equations in power series, and by the method of expansion in powers of a parameter of Poincaré-Liapunov. He finally suggests that these or similar techniques will prove useful in transforming computational schemes, transforming logical and set-theoretical formulas, analysis and synthesis of relay-contact schemes, various computations and transformations in algebra and topology, and formal analysis of deductive theories. "Along with use in approximate computations in applied mathematics, machine analysis must obtain systematic application in the family of so-called theoretical regions of mathematics." *J. W. Carr, III* (Ann Arbor, Mich.)

4361:

✱Müller, Paul Friedrich. Die Integrieranlage des Rheinisch-Westfälischen Instituts für Instrumentelle Mathematik in Bonn. Forschungsberichte des Wirtschafts- und Verkehrsministeriums Nordrhein-Westfalen, Nr. 310. Westdeutscher Verlag, Köln und Opladen, 1956. 54 pp. DM 14.45.

A mechanical differential analyser was constructed by the firm of Schoppe and Faeser, Munich, and installed in the Institut für instrumentelle Mathematik in Bonn, about July 1954. This article is a summary of differential analyser methods with particular attention to the operation of this machine.

No details of construction are given, and there appears to be nothing novel in the presentation. There are reproductions of solutions by the Bonn analyser of several equations, such as those of the Legendre polynomials 1 to 10, and other special systems.

A bibliography of 79 references on differential analyser applications may be of interest to the user of an analyser. *G. R. Stibitz* (Cambridge, Vt.)

MECHANICS OF PARTICLES AND SYSTEMS

See also 4038, 4066, 4422.

4362:

Blaschke, Wilhelm. Sull'uso dei quaternioni duali nella cinematica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 291–293.

Se una retta è definita mediante un sua punto x ed il vettore unitario della sua direzione r e se $\bar{r}$ significa il prodotto vettoriale $\bar{r}=x\times r$, la retta è caratterizzata dal vettore unitario duale $\mathbf{r}=r+\varepsilon\bar{r}$, $\varepsilon^2=0$. L'influsso d'un movimento spaziale sulla rette si può rappresentare mediante un quaternione duale $\mathbf{Q}=Q+\varepsilon\bar{Q}$ con $\mathbf{r}=\tilde{\mathbf{Q}}\mathbf{r}'\mathbf{Q}$. Maggiori particolarità si trove nella note dell'autore Ann. Acad. Sci. Fenn. Ser. A. I. no. **250**/3 (1958) [MR **19**, 1099]. *O. Bottema* (Delft)

4363:

Chicarro, Mateo F. Trajectories of a material point in the plane. Gac. Mat., Madrid, **10** (1958), 13–25. (Spanish)

An exposition, chiefly of some of the remarks of Darboux on the analogies between dynamics and the theory of geodesic curves on a surface.

4364:

Grindei, I. Sur l'équivalence des systèmes mécaniques non holonomes. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) 3 (1957), 197–206. (Russian and Romanian summaries)

Considered are non-holonomic systems in n Lagrangian variables x^i, $i=1, 2, \cdots, n$, with constraints given by equations of the form $f_\sigma(x^1, \cdots, x^n, dx^1/dt, \cdots, dx^n/dt, t)=0$, $\sigma=m+1, \cdots, n$, or of the form $B_{\sigma i}d^2x^i/dt^2+B_{\sigma 0}=0$, where the B are given functions of the x, $\dot{x}=dx/dt$, and t. It is shown that, by adjunction of certain forms depending on the $\dot{x}^i$, such systems can be reduced to holonomic systems which admit the constraint equations as integrals. This generalizes a result of C. Agostinelli [Boll. Un. Mat. Ital. (3) **11** (1956), 1–9; MR **17**, 1246]. Then the problem of the equivalence of two non-holonomic systems with constraints $f_\sigma(x, \dot{x}, t)=0$ is reduced to the equivalence problem of two spaces of affine connection [see S. S. Chern, Bull. Sci. Math. France **63** (1939), 206–212; MR **1**, 145; G. Vranceanu, Bull. Math. Soc. Roumaine Sci. **42** (1940), no. 2, 69–97; MR **7**, 34]. *D. J. Struik* (Cambridge, Mass.)

4365:

Bottema, O. On the motion of a particle on a torus. Nieuw Arch. Wisk. (3) **5** (1957), 75–80.

One considers the motion of a particle on the torus (S) generated by the revolution of a circle (r) about an axis in its plane (R) under a central force $f(\rho=R+r\sin\theta)$. The torus has the three systems of circles: the parallel circles (θ=const), the meridional circles (φ=const) and the Villarceau circles, which are the intersection of S with a bitangential plane ($z=y\operatorname{tg}\alpha$, $\sin\alpha=rR^{-1}$). It is shown that if and only if the force is inversely proportional to the third power of distance, the Villarceau circles are possible orbits too. Further, the author shows that the projection of such motion on the Oxy plane is the same as the motion of a particle under Newton's law with origin as the centre of attraction, while the projection of this circle on this plane is an ellipse. The motion along these circles under this force is unstable. *D. P. Rašković* (Belgrade)

4366:

Rionero, Salvatore. Sul principio dell'effetto giroscopico. Ricerche Mat. **7** (1958), 14–20.

As used here, the term principle of the gyroscopic effect refers to a method for obtaining approximations to solutions of problems concerning a rigid body rotating about a fixed point. For details see a paper by Stoppelli [Giorn. Mat. Battaglini (4) **4(80)** (1950–51), 14–38; MR **12**, 760]. In the present paper the accuracy of the approximation is considered, and an example is discussed in which the accuracy is unsatisfactory. *L. A. MacColl* (New York, N.Y.)

4367:

Tchebotarev, G. A. A symmetric orbit of a rocket for flight around the moon. Gaz. Mat., Lisboa, **19** (1958), 1–7. (Portuguese)

Expository.

4368:

Teixeira, Gaspar J. The creation of an artificial satellite of the earth. Gaz. Mat., Lisboa, **18** (1957), 9–16. (Portuguese)

An exposition of the relevant parts of rational mechanics.

4369:

Chiara, Luciano. Sull'equazione fondamentale della dinamica del punto di massa variabile. Atti Accad. Sci. Lett. Arti Palermo. Parte I (4) **16** (1955/56), 169–177 (1957).

4370:

Chiara, Luciano. Casi in cui nel problema dei due corpi di massa decrescente l'eccentricità varia in ragione inversa della massa. Atti Accad. Sci. Lett. Arti Palermo. Parte I (4) **16** (1955/56), 179–192 (1957).

4371:

Leitmann, G. Stationary trajectories for a high-altitude rocket with drop-away bolster. Astronaut. Acta **2** (1956), 119–124.

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 4332.

4372:

Waldmann, L. Die Boltzmann-Gleichung für Gase aus Spinteilchen. Z. Naturf. **13** (1958), 609–620.

Let us contemplate a gas where the individual particles have an angular momentum. We shall assume that the gas is so dilute that only binary collisions take place; the translational motion of the particles can be handled classically except during collisions; during collisions the magnitude of the angular momentum does not change, while its orientation may; if external fields are present their gradients are small within distances of the order of the thermal de Broglie wave length and exert no torque. Then, if a particle in the system has a fixed total angular momentum J, its state can be characterized by a distribution matrix $f_{MN}(r, c, t)$, where M, N range from $-J$ to $+J$, r is the position of the particle, c its velocity at time t. The matrix is so defined that the expectation value of an operator O at r is given by $\int \mathrm{Tr}(Of)dc$, the integration being performed over all velocities.

The author now proceeds in the traditional way to derive the Boltzmann equation. The change of the expectation value of an operator will consist of the usual streaming terms and the change due to collisions. The streaming terms have the same structure here as for a gas with no angular momentum; the collision terms can be computed as soon as the scattering amplitudes are known. The resulting equation admits as conserved quantities the energy, momentum, and number of particles, but not the angular momentum. The collision terms in the equation still give the difference between the number of particles created in a given state and the number of particles destroyed in that given state; however, the structure of these terms is not the same as for particles with no angular momentum, since it involves the scattering amplitudes and not only the scattering cross section. If the direction of the angular momenta is distributed isotropically the scattering amplitudes combine into a scattering cross section and the equation reduces to the usual Boltzmann equation. *N. L. Balazs* (Chicago, Ill.)

4373:

Fixman, Marshall. Theory of diffusion near walls. J. Chem. Phys. **29** (1958), 540–545.

The random force in the Langevin equation of motion of a molecule in the liquid is identified with the force due to the excess pressure of a spherical acoustical wave which carries momentum. Using this model the hindering effect of a wall on diffusion is evaluated, considering both specular reflection and diffuse reflection at the wall.

D. ter Haar (Oxford)

4374:

Collins, Frank C.; and Raffel, Helen. Statistical mechanical theory of transport processes in liquids. J. Chem. Phys. **29** (1958), 699–710.

The irreversibility which produces the driving force for transport processes is introduced by expressing the time derivatives of the non-equilibrium stresses in the liquid in terms of the gradient of the properties which are being transported. It is then unnecessary to introduce a time-smoothing. The resulting equations are used to find expressions for the hydrodynamic frictional and self-diffusion coefficients, the viscosity and the thermal conductivity. For the case of a Lennard-Jones potential the viscosity and thermal conductivity are evaluated and the resulting values agree reasonably well with experimental data for liquid argon. *D. ter Haar* (Oxford)

4375:

Peretti, Jean. Propriétés combinatoires de la fonction $\mathfrak{F}(z)$ attachée à un réseau. C. R. Acad. Sci. Paris **246** (1958), 3310–3312.

The function in question is one discussed previously by the author [same C. R. **241** (1955), 461–463, 544–546; MR **17**, 568] in relation to the elastic vibration spectra of three-dimensional crystal lattices. Here it is shown that this function is also the generating function for (1) the number of oriented polygons on a cubic lattice and (2) the probability that a random walk returns to its origin after n steps. *G. Newell* (Stockholm)

4376:

Ayres, R. U. Variational approach to the many-body problem. Phys. Rev. (2) **111** (1958), 1453–1460.

The Löwdin-Mayer reduced density matrix technique is used to discuss a system of interacting fermions. Only one- and two-particle reduced density matrices are considered, and it is shown that the requirement that these can be obtained by variational methods eliminates all non-physical reduced density matrices obtained by other authors. *D. ter Haar* (Oxford)

4377:

Barut, A. O. Covariant quantum statistics of fields. Phys. Rev. (2) **109** (1958), 1376–1380.

Landau's theory of multiple production [Izv. Akad. Nauk SSSR Ser. Fiz. **17** (1953), 51–64] requires a statistical mechanics of relativistic fields [see also M. Namiki and C. Iso, Progr. Theor. Phys. **18** (1957), 591–613; MR **19**, 1233]. Here a statistical mechanical formalism is set up, starting from the partition function

$$\mathrm{Tr}\exp(\alpha Q - \beta_j P^j + \lambda_{ij} J^{ij} + \cdots).$$

Q is the operator of total charge, P^j the total energy momentum four-vector, J^{ij} the total angular momentum,

and the dots refer to any other integrals of the motion that may exist. This partition function is evaluated for uncharged particles without mass, in which case the average energy satisfies the Stefan-Boltmann law, as it should. For the case of charged particles with mass the average energy, charge and momentum are expanded in powers of the fugacity e^{α}. It is also shown that, if the particles are able to exchange charge, the statistics are materially altered. Finally the case of non-vanishing average spin is considered.

The author argues that the statistical properties are not affected by the presence of interaction between the fields, as long as there are no bound states. The reason is that the operators Q, P^j, J^{ij}, $\cdots$ for interacting fields are connected with those for free fields by a unitary transformation, which does not affect the trace. This argument is hard to believe, in view of the existence of non-ideal gases. *N. G. van Kampen* (Utrecht)

ELASTICITY, PLASTICITY

4378:

Prager, William. Elastic solids of limited compressibility. Proc. Int. Congr. Appl. Mech. Brussels, 1956, 205–211.

The usual law for classical isotropic elasticity involving two elastic constants, the shear modulus and the bulk modulus, is postulated for a body, as long as the density remains below a given critical value. At the pressure that corresponds to the critical density the solid is assumed to become incompressible; for higher pressures the stress strain law contains only one elastic constant, the shear modulus. The general boundary value problem for this type of solid is formulated, the uniqueness of the solution is discussed, and associated extremum principles are established. *A. E. Green* (Newcastle-upon-Tyne)

4379:

Miyamoto, Hiroshi. On the problem of the theory of elasticity for a region containing more than two spherical cavities. Bull. JSME 1 (1958), 103–108.

The author considers the stress distribution in an infinite region bounded internally by a number of spherical cavities whose centres are collinear. The method of solution is only developed in detail for the case of two spherical cavities under axisymmetric loading. This special case was considered previously by Sternberg and Sadowsky [J. App. Mech. **16** (1952), 19–27; MR **14**, 926] by a different method. Some numerical results are given. *A. E. Green* (Newcastle-upon-Tyne)

4380:

Nicolovius, Rüdiger. Abschätzung der Lösung der ersten Platten-Randwertaufgabe nach der Methode von Maple-Synge. Z. Angew. Math. Mech. **37** (1957), 344–349. (English, French and Russian summaries)

For the biharmonic boundary value problem

$$\Delta\Delta w = p(x, y) \text{ in } B, \quad w = f(s), \quad \partial w/\partial n = g(s) \text{ on } \Gamma,$$

where B is a plane domain and Γ its boundary, the author uses the Prager-Synge hypercircle method [cf. J. L. Synge, Proc. Roy. Soc. London. Ser. A. **191** (1947), 447–467; MR **10**, 81; and #4073 above] with the scalar product

$$(\varphi, \psi) = \tfrac{1}{2} \iint \Delta\varphi\Delta\psi dxdy.$$

To bound the solution at a point, he uses the method of C. G. Maple [Quart. Appl. Math. **8** (1950), 213–228; MR **12**, 704]. He carries out numerical calculations for the case where B is the part of a circular area contained between two parallel equal chords and $p=1$, $f=g=0$, using the orthonormalization process of M. O. Peach [Bull. Amer. Math. Soc. **50** (1944), 556–564; MR **6**, 52] and obtaining the value of w at the centre with an error of less than 8.6 per cent. *J. L. Synge* (Dublin)

4381:

Mazzarella, Franco. La piastra quadrata appoggiata al centro. Giorn. Mat. Battaglini (5) 5 (85) (1957), 188–196.

A formal solution is given of the problem of a flat, square, elastic plate with free edges; the plate is uniformly loaded over one face and is supported by an isolated force at the centre. Let the transverse displacement be $w = w_0 + w_1$. Then w_1 is chosen as the simplest possible (singular) solution of the inhomogeneous plate equation; w_1 does not satisfy the free edge boundary conditions. Next w_0, a solution of the homogeneous plate equation expressed as a linear combination of potential-type functions, is determined so that w_0+w_1 satisfies the boundary conditions. These conditions lead to an infinite system of linear equations in certain unknown coefficients entering into the specification of w_1. The solution for w thus found is formal, because in practice it is only possible to consider an approximate solution corresponding to a finite number of these coefficients, as the author himself indicates. *H. G. Hopkins* (Fort Halstead)

4382:

Egerváry, J. Begründung und Darstellung einer allgemeinen Theorie der Hängebrücken mit Hilfe der Matrizenrechnung. Magyar Tud. Akad. Mat. Kutató Int. Közl. **2** (1957), 3–32. (Hungarian. German summary)

Mehrere neuere Aufsätze über die Berechnung der Hängebrücken lassen eine Tendenz erkennen, infinitesimalen Operationen zu vermeiden. Die Differentialquotienten werden dabei durch Differenzenquotienten, die stetig verteilte lebende Last durch Knotenlaste ersetzt.

In der vorliegenden Arbeit wird eine finite Theorie der Kettenbrücken aufgebaut. Bei konsequenter Verwendung der Clapeyronschen Gleichungen der Balkentheorie und bei Annäherung der eventuell stetig verteilten lebenden Last durch Einzelkräfte (Knotenlaste) werden zur sachgemässen Behandlung der auf diese Weise entstehenden linearen Gleichungen matrizentheoretische Hilfsmittel herangezogen. Als Grundgleichung einer Kettenbrücke ergibt sich eine Matrizengleichung welche als Unbekannte die Durchbiegungsmatrix des Versteifungsträgers enthält. Zur Auflösung dieser Matrizengleichung ist nur das Invertieren von Kontinuantenmatrizen notwendig, wofür einfache, auch maschinell gut durchführbare Rechenschemata angegeben werden. Bei einer Kettenbrücke mit $n-1$ Hängestäben erhält man für die Zunahme der Horizontalspannung eine Bestimmungsgleichung n-ten Grades.

Bei einem Grenzübergang, wobei die Anzahl der Hängestäbe unbegrenzt zunimmt, geht die vorher erwähnte Matrizengleichung in die wohlbekannte Melansche Differentialgleichung der Hängebrücken über.

Im letzten Paragraph werden diejenigen Vereinfachungen besprochen, welche sich bei einer gleichmässigen Kettenbrücke einstellen.

{Diese Arbeit ist dem Inhalt nach identisch mit der deutschsprachigen Arbeit des Verfassers, Abh. Internat. Verein. für Brücken- und Hochbau 16 (1956); 149–184.}

Zusammenfassung des Autors

4383:

Chakraborty, Sakti Kanta. Propagation of waves in isotropic elastic medium generated by forces on the inner surface of a nearly spherical cavity. Bull. Calcutta Math. Soc. 49 (1957), 207–215.

4384:

✱Leaderman, Herbert. Viscoelasticity phenomena in amorphous high polymeric systems. Rheology: theory and applications. Edited by F. R. Eirich. Vol. 2, pp. 1–61. Academic Press Inc., New York, 1958. $18.00.

This survey of analytical representations of linear visco-elastic responses and of their relations to the underlying physical phenomena is up-to-date and comprehensive. Unfortunately from the mathematical point of view, it is concerned essentially with one-dimensional formulation of the relations between stress and strain and their time-derivatives, even when discussing the effect of such a phenomenon as bulk-viscosity. While this may be a deliberate limitation in order to increase the readability and usefulness of the article for the less mathematically inclined readers of this handbook, who probably constitute the majority of such readers, it is nevertheless a loss to the readers strongly interested in the mathematical aspect of the subject, whose interest might therefore be better served by the recent summary of Staverman and Schwarzl, of the same subject in the first chapter of Vol. IV of H. A. Stuart's "Physik der Hochpolymeren".

Apart from this objection to the selected approach, the article clearly shows the author's wide knowledge of the subject, particularly of the interrelation between molecular processes and phenomenological behavior. It is divided into five parts dealing, respectively, with creep and delayed elasticity, stress-relaxation and vibration, mechanical models of 3, 5 and 6 elements, continuous spectra and the superposition principle, including a discussion of the interrelation between creep and relaxation, as well as of the approximate determination of spectra of relaxation or retardation times, and, finally, with the interrelation of phenomenological visco-elastic response and molecular structure, including a short discussion of non-linearity.

Considering the very wide range of subjects covered in this survey the list of selected references, which should substantially add to the value of the article by providing the interested but uninformed reader with an important source for further study beyond the necessarily limited coverage of the survey, is disappointingly scant. Basic contributions such as those of Thompson, Becker, Kuhn and collaborators, Burgers, Simha, Zener and others are not mentioned, nor are indications given of any Russian work in this field. *A. M. Freudenthal* (New York, N.Y.)

4385:

✱Rzhanitsyn, A. R. The shape at collapse of elastic-plastic plates simply supported along the edges. Tech. Rep. No. 19, Office of Naval Research, Contract Nonr-562 (10) NR-064-406, Division of Applied Mathematics, Brown University, Providence, R. I., 1957. i+47 pp.

[Translated from the Russian: Chap. 8, Structural analysis taking account of plastic properties, 2d ed., Moscow, 1954.] In recent years, there has been much interest in the development and application of theories appertaining to yield-point mechanisms and loads for flat plates of non-hardening plastic-rigid material. Such yield-point mechanisms do not necessarily correspond to actual collapse loads, and their progressive development may well be restricted and modified in practice by middle surface forces arising in the course of geometrical changes. Although the development of the basic theory for any assigned yield criterion (involving only the stress moments) is reasonably straightforward, in applications, at least to ductile metal plates, attention has largely been confined to problems of circular plates under rotationally-symmetric conditions. Under other conditions, applications are still severely limited by the seemingly complex nature of the governing equations, these involving the stress moments and curvature rates. It is known that elliptic or mixed elliptic-parabolic equations generally occur when the von Mises or Tresca yield criterion is adopted. Now inasmuch as the solution of particular problems will normally involve the determination of initially unknown boundaries separating regions of plates in differing plastic regimes, it is clear that the Tresca yield criterion may involve simpler mathematical circumstances than does the von Mises yield criterion. It is natural to inquire if a yield criterion exists for which the equations are generally parabolic only. This simpler situation does in fact hold for the case of the Johansen "square" yield criterion. This latter criterion may be regarded in either one of two ways, viz., as directly applicable to doubly-reinforced concrete plates and as an approximation to both the Tresca and the von Mises yield criteria, themselves directly applicable to ductile metal plates. Importantly, the equations appropriate to the Johansen yield criterion allow certain simple types of discontinuity, e.g., in particular, discrete and continuous distributions of hinge lines (or fracture lines), to characterize possible yield-point mechanisms. As the present reviewer has pointed out in a paper published in Proc. 9th Internat. Congress Appl. Mech., Brussels, 1956, this fact appears to provide the essential mathematical basis for the so-called fracture line theory of K. W. Johansen [Fracture line theory, Gjellerup, Copenhagen, 1943]. Although this theory has been developed specifically in connexion with reinforced concrete plates, it seems quite likely that the theory will be of additional importance in extended application as an approximation to the Tresca and von Mises theories for ductile metal plates. In applications of Johansen's theory, the inherent simplicity of procedure depends upon the supposition of quite simple yield-point mechanisms, these being admitted by the equations due to their parabolic nature. Assuming a definite type (or class) of possible yield-point mechanisms, upper bounds for yield-point loads may be obtained in the usual way. However, it is not always easy to say definitely whether or not actual values of yield-point loads have in fact been found. In certain simple problems, the results may be proved exact by finding the complete solution, but generally, even when "physical intuition" suggests strongly that the results are exact (or, more probably, not seriously in error), there can be no proof so long as the complete solution remains unknown.

Rzhanitsyn, essentially adopting the Johansen fracture line theory, gives analysis leading to the determination of upper bounds for yield-point loads of flat plates. This work apparently stems from previous Russian work of A. A. Gvozdev and others, no direct reference being made to the work of Johansen. A considerable variety of circumstances is envisaged, and the different types of problems considered are too many and diverse to be summarized in detail here. The plates are often of quite arbitrary convex shape, and detailed results are given for

many interesting special cases of circular, elliptic and polygonal plates; the plates are supported at and within their edges, the support being continuous and at isolated points; the boundary conditions are simple support, built-in and free; and although attention is given primarily to single concentrated loads, some brief attention is given also to uniformly-distributed loads. The yield-point mechanisms vary a great deal according to the problem considered but can involve discrete and continuous systems of rectilinear hinge lines and curved peripheral hinge lines, the principal curvature rate along any particular rectilinear hinge line of course being zero. The yield-point mechanisms sometimes involve certain arbitrary features, these to be determined in the course of analysis by the appropriate minimal methods, and need not necessarily extend over the entire plate. In the case of a concentrated load, hinge lines radiate from the point of application of load. The simplicity of many of the final formulae for yield-point loads is very noticeable.

According to the translator, R. M. Haythornthwaite, the translation was prepared because a modern statement of fracture line theory, including the Russian developments, did not appear to be available in English. The present translation of Rzhanitsyn's work is to be welcomed; it will provide a wider audience with immediate access to recent Russian work on this subject. In independent work, E. H. Mansfield [Proc. Roy. Soc. London Ser. A **241** (1957), 311–338; MR **19**, 597] has also recently given detailed analysis of a wide variety of similar problems. Mansfield has thereby covered very much the same ground as Rzhanitsyn, but has included some attention to cases of concave plates and of combined loads.

H. G. Hopkins (Sevenoaks)

STRUCTURE OF MATTER

See also 4384.

4386:

Yosida, Kei. Remarks on the theory of superconductivity. Phys. Rev. (2) **111** (1958), 1255–1256.

The theories of superconductivity developed by J. Bardeen, L. N. Cooper, and J. R. Schrieffer [Phys. Rev. (2) **108** (1957), 1175–1204; MR **20** #2196] and by N. N. Bogoliubov [Soviet Physics. JETP **34**(7) (1958), 41–46 (58–65 of Russian original; MR **20** #5670a)] both predict the existence of a finite interval between the ground state and the first excited state of the electrons in interaction with the lattice vibrations. The author shows that, despite apparent differences, the two theories are essentially the same, and can be transformed into each other by certain unitary transformations. *E. L. Hill* (Minneapolis, Minn.)

FLUID MECHANICS, ACOUSTICS

See also 4384.

4387:

Sabaneev, V. S. On the motion of an ellipsoid in a fluid bounded by a plane wall. Vestnik Leningrad. Univ. **13** (1958), no. 13, 132–146. (Russian. English summary)

The paper considers the motion of an ellipsoid in an ideal incompressible fluid bounded by a plane wall. Using Eisenberg's method [J. Appl. Mech. **17** (1950), 154–158; MR **12**, 58], the author gives approximations to the velocity potentials for all cases where the semimajor axis of the ellipsoid keeps parallel to the wall all through the motion. *From the author's summary*

4388:

Roy, Maurice. Remarques sur l'écoulement tourbillonnaire autour des ailes en flèche. Z. Angew. Math. Phys. **9b** (1958), 554–569.

The author presents, in a descriptive manner, some ideas about the flow field near the leading edge of a highly swept wing at incidence. He explains how the flow separation differs from the bubble type found in two dimensions and describes the configuration of limiting streamlines, attachment lines and separation lines which arises. An examination of the boundary conditions along a separation line shows that a difficulty occurs unless the vortex sheet springing from the line is initially tangential to the wing surface. For the conical flow past a delta wing a two-dimensional "pseudo-courant" is introduced in which, for an incompressible main flow, the condition of continuity in the cross-flow plane is violated. The author describes how the sink distribution needed to restore this can be condensed onto the section of the vortex sheet by the cross-flow plane. A type of secondary separation which alternates between a bubble and a vortex sheet is described. The observed periodic variation along the leading edge of the limiting streamline pattern is attributed to the tearing off of the vortex sheet at one station, with the formation of a second sheet downstream, which is torn off in turn. The author concludes that these complicated phenomena can only be described by a development of boundary layer theory.

J. H. B. Smith (Farnborough)

4389:

Rundgren, Lennart. Water wave forces. A theoretical and laboratory study. Kungl. Tekn. Högsk. Handl. Stockholm, no. **122** (1958), 123 pp.

Of considerable experimental but little theoretical interest. *F. Ursell* (Cambridge, England)

4390:

Meksyn, David. The boundary-layer equation for axially symmetric flow past a body of revolution: motion of a sphere. J. Aero. Sci. **25** (1958), 631–634.

"The equations of the boundary layer for an axially symmetric flow past a solid of revolution are reduced to a form identical with that derived by the author for plane flow. This method is applied to the case of a flow past a sphere, and the frictional force and the separation point are evaluated.

The equation is integrated asymptotically by the method developed for plane flow, where the partial differential equation is reduced to an ordinary equation. A new straightforward method of step-by-step integration of this equation is used." (Author's summary)

C. C. Lin (Cambridge, Mass.)

4391:

Ray, M. Boundary layer in a perfect gas over a flat plate under pressure gradient. Bull. Calcutta Math. Soc. **49** (1957), 133–138.

Approximate solution of the two dimensional boundary layer equations for the case where the external pressure p, the enthalpy i, density ρ and viscosity μ do not deviate much from the constant values p_0, i_0, ρ_0, μ_0. Two ordinary differential equations for velocity u (along the wall) and

density variation ρ' are obtained with x (coordinate along the wall) and stream function ψ as independent variables. The integration is simple, but the solutions are deficient in two points: 1) $\partial u/\partial y$ is finite at the outer edge of the layer; 2) $\partial u/\partial y$ is infinite at the wall.

I. Flügge-Lotz (Stanford, Calif.)

4392:

Aksenov, A. P. The laminar boundary layer of a cone in supersonic flow. Vestnik Leningrad. Univ. **12** (1957), no. 13, 113–128. (Russian. English summary)

In this paper the problem of the determination of surface temperature and skin friction on cones is solved by using integral relations of momentum and energy, both when heat radiation is neglected and when the heat phenomenon is taken into account. The Prandtl number is assumed to be a function of temperature only and viscosity is assumed to vary according to Sutherland's formula. *Author's summary*

4393:

Lessen, M. On the hydrodynamic stability of curved laminar flows. Z. Angew. Math. Mech. **38** (1958), 95–99. (German, French and Russian summaries)

4394:

Miles, John W. On the disturbed motion of a plane vortex sheet. J. Fluid Mech. **4** (1958), 538–552.

The problem of the stability of a plane vortex sheet in an inviscid fluid which separates two parallel streams is formulated as an initial value problem and treated by transform methods. A detailed analysis is made for the case in which the initial displacement of the surface is zero but its initial transverse velocity is spatially periodic. The resulting eigenvalue equation then yields the condition for stability as a function of the relative velocities of the two streams; in particular, it is found that unstable supersonic disturbances exist for

$$a_+ + a_- < |U_+ - U_-| < (a_+^{2/3} + a_-^{2/3})^{3/2},$$

where $U_\pm$ are the velocities of the two streams and $a_\pm$ the corresponding sonic velocities.

W. H. Reid (Providence, R.I.)

4395:

Gorelov, D. N. Oscillating airfoil in subsonic flow. Vestnik Leningrad. Univ. **12** (1957), no. 13, 93–101. (Russian. English summary)

The article contains an approximate solution of the problem of the oscillating airfoil in subsonic flow. This solution is found to agree with the exact solution at small Struhal numbers. *Author's summary*

4396:

Manwell, A. R. On the breakdown of plane transonic flow. Proc. Roy. Soc. London. Ser. A. **245** (1958), 481–520.

The breakdown of steady two-dimensional transonic flow is discussed by analyzing the singularities on the sonic line. The author studies the solution of the boundary value problem for the perturbation stream function, i.e., for the difference between the stream functions of two flows which satisfy almost the same conditions on the subsonic part of the boundary and an appropriate portion of the supersonic part. In particular, he considers the free jet problem where the perturbation stream function satisfies the Tricomi equation $s\phi_{\theta\theta}+\phi_{ss}=0$, with $s\phi_\theta ds-\phi_s d\theta$ prescribed on the boundary, where $s>0$ and ϕ is prescribed on the characteristic for $s<0$. This problem is well set, unlike the general transonic problem. The author demonstrates explicitly, for special cases, that the acceleration has a singularity at the intersection of the sonic line and the boundary away from the edge of the jet, i.e., on the central streamline. It is not clear to the reviewer why there should not be an actual singularity in the acceleration at such a point, but the result clearly demonstrates that the perturbation is not valid there.

The second part of the paper contains a proof of the existence of a solution to the above boundary value problem with a suitable choice of the elliptic boundary. The author uses the methods of Tricomi through which it is possible to display the singularities, rather than the indirect methods of Germain. The existence of the solution has also been announced by S. Agmon [Atti de Convegno Internazionale sulle Equazioni alle Derivate Parziali, Trieste, 1954, pp. 54–68. Ed. Cremonese, Roma, 1955; MR **17**, 859]. *C. S. Morawetz* (New York, N.Y.)

4397:

Kopylov, G. N. Aerodynamic characteristics of thin wedge-shaped profiles in transonic flows. J. Appl. Math. Mech. **22** (1958), 182–191 (133–138 Prikl. Mat. Meh.).

Ce travail a pour sujet l'étude d'un écoulement transsonique autour d'un profil losangique, dans la région située en aval de la ligne sonique issue d'un sommet du losange. Habituellement, on traite la région située en aval de la frontière transsonique par la méthode des caractéristiques; l'auteur propose une solution analytique de ce problème dans le cadre de la théorie simplifiée des écoulements transsoniques. Dans le plan de l'hodographe, la fonction de courant est solution d'une équation de Tricomi. La formule fondamentale utilisée dans cette étude est celle permettant le calcul d'une solution de l'équation de Tricomi dont la dérivée normale le long d'un segment de la ligne parabolique est connue et qui s'annule le long d'une caractéristique-formule donnée par P. Germain et R. Bader [O.N.E.R.A. Publ. no. **54** (1952); MR **14**, 654; formule (32)]. Les formules permettant le passage au plan physique sont exprimées avec des intégrales relativement simples à calculer. Le travail est poussé jusqu'au calcul numérique pour divers régimes d'écoulements transsoniques. La comparaison des résultats obtenus avec certains résultats expérimentaux semble fort satisfaisante. *P. Germain* (Paris)

4398:

Pridmore-Brown, D. C. Sound propagation in a fluid flowing through an attenuating duct. J. Fluid Mech. **4** (1958), 393–406.

"A study is made of the propagation of sound in both a constant gradient shear flow and a turbulent shear flow above a flat surface. Curves are presented showing how, in the case of downstream propagation, the flow gradient tends to channel the sound energy into a narrow layer next to the wall. These results are used in estimating the effect of a flow on the attenuation of sound in a duct with absorbing side walls." (Author's abstract)

P. M. Morse (Cambridge, Mass.)

4399:

Ivanovskiĭ, A. I. On the connection of acoustical streaming with sound absorption. Akust. Ž. **4** (1958), 143–152. (Russian)

Three differential equations of the same form, but with different coefficients, are derived for streaming in an acoustic field using (1) first and second order approximations in the Navier-Stokes equation, (2) a linear visco-

elastic approximation to the medium, (3) the Boltzmann equation for hereditary media. The latter explain the frequency-dependence of the stream velocity.

W. W. Soroka (Berkeley, Calif.)

4400:

Kleiman, Ya. Z. Propagation of weak discontinuities in a multicomponent medium. Akust. Ž. 4 (1958), 253–262. (Russian)

Considering a multicomponent medium consisting of N cells containing different media, which may be moving, the equations of wave propagation in each pure medium and the kinematic conditions of compatibility in the multicomponent medium are used to derive an algebraic equation of degree $2N$ giving, in general, $2N$ values of the propagation velocity of a weak disturbance. In particular, a two-component medium is discussed.

W. W. Soroka (Berkeley, Calif.)

4401:

Lapin, A. D. Scattering of sound waves in irregular wave guides. Akust. Ž. 4 (1958), 267–274. (Russian)

Treated as a two-dimensional problem, a normal sound wave in the smooth homogeneous left half of an infinitely long acoustic wave guide is incident on the right half, which has either (1) a statistically inhomogeneous medium, or, (2) one rough wall. The scattered field, which is not small in relation to the incident field, is considered as a sum of propagation modes identified with a smooth, homogeneous wave guide.

W. W. Soroka (Berkeley, Calif.)

4402:

Schenk, J.; and Van Laar, J. Heat transfer in non-Newtonian laminar flow in tubes. Appl. Sci. Res. A. 7 (1958), 449–462.

The authors examine the heat transfer from fluids in steady laminar flow through circular cylindrical tubes. To account for non-Newtonian effects a radial velocity profile suggested by Prandtl and Eyring is assumed. The energy equation in cylindrical polar coordinates is solved numerically for various values of the physical constants, and a comparison is given with the results obtained by other workers. In these calculations, the longitudinal conduction and heat dissipation terms are neglected. Some discussion of the effect of heat dissipation is, however, given in an appendix. The influence of thermal resistance of the tube wall in the radial heat conduction problem is also examined. *J. E. Adkins* (Nottingham)

4403:

Lin, C. C. Note on a class of exact solutions in magneto-hydrodynamics. Arch. Rational Mech. Anal. 1 (1958), 391–395.

Finite conductivity and non-zero kinematic viscosity are assumed. The solutions of the magneto-hydrodynamic equations are obtained in terms of indeterminate functions of one linear space-coordinate and the time, under the assumption that the velocity field, the magnetic field and the pressure-gradient depend linearly on the other two space-coordinates. Fifteen functions are introduced by this method and the sixteen equations they must satisfy are shown to form a consistent set.

G. C. McVittie (Urbana, Ill.)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 3774, 3775, 4184, 4438.

4404:

★Longhurst, R. S. Geometrical and physical optics. Longmans, Green and Co., London-New York-Toronto, 1957. xvi+534 pp. $12.00.

This book, as the author states in the preface, is intended primarily as a text for undergraduates reading for a first degree in physics. In this respect it admirably fulfills its purpose.

The subject matter (with some important exceptions) and the arrangements of the topics follow the traditional line of the intermediate textbooks in optics. Starting with the elementary laws of reflection and refraction, it includes such topics as geometrical optics of simple lens systems, of prisms and of spectroscopic instruments, wave properties of light, interference phenomena including multiple beam interferometry, Huygen's principle and its formulations by Fresnel and Kirchhoff, diffraction by simple apertures, gratings, etc., Fresnel and Fraunhofer diffraction phenomena, image formation of optical systems, geometrical aberrations of third order, physiological optics, a brief account of electromagnetic theory of light dispersion, and absorption and crystal optics. The final chapter contains a short but clear discussion of the various methods and apparatus employed for measurement of the velocity of light.

To the reviewer the main features of the book are: a rather extensive and well illustrated discussion of the principal instruments employed in optical experiments and scientific research; the chapter dealing with multiple beam interferometry and the topics on image formation; and the elementary treatment of geometrical aberrations of third order, which are either ignored or briefly treated in even recent books of the same level as the present one. Another good feature is the large number of excellent illustrations. On the other hand, the sections on crystal optics are rather too descriptive to give a clear idea of the optical phenomena in crystals, and topics such as photoelectric effect, Compton effect, Čerenkov radiation, Zeeman and Stark effects, Raman effect and introduction of x-ray optics, which are based on the concept of quanta, are left out or very briefly mentioned, although the author has discussed some instruments which are founded on the principles of these phenomena. Another drawback is the lack of problems. In spite of these omissions, the book should prove very useful to both students and teachers, and to scientists working in experimental optics. Although there is no lack of good books of intermediate level, the book under review can be well recommended for a one or two semester course for undergraduates in our colleges and universities.

N. Chako (Flushing, N.Y.)

4405:

Fedorov, F. I. Invariant methods in the optics of transparent non-magnetic crystals. Akad. Nauk SSSR. Kristallografiya 3 (1958), 49–56. (Russian)

Basic relations in the optics of transparent non-magnetic crystals are obtained by an invariant method. This avoids the use of the principal axes of the specific inductive capacitance tensor. This method was previously developed by the author [Akad. Navuk BSSR Trudy Inst. Fiz. Mat. **1957**, no. 2, 230]. It is indicated that the treatment may be extended to the most general case of absorbing crystals. *J. E. Rosenthal* (Passaic, N.J.)

4406a:

Stepanov, K. N.; and Tkalič, V. S. The vibrations of an electron plasma in external electric and magnetic fields. Ž. Tehn. Fiz. **28** (1958), 1789–1800. (Russian)

4406b:

Stepanov, K. N.; and Tkalich, V.S. On the oscillations of an electron plasma in external electric and magnetic fields. Soviet Physics. Tech. Phys. **28** (3) (1958), 1649–1659.

A study is made of the propagation of electromagnetic waves in a plasma placed in crossed electric and magnetic fields. The thermal motion of the electrons is taken into account. It is assumed, however, that the vibration frequency of the plasma is sufficiently high to permit neglecting ion motion. The state of the plasma is described by the electron distribution function. The dispersion equation is derived and studied. The case where rot **E** $\neq 0$ is also taken into account.

J. E. Rosenthal (Passaic, N.J.)

4407:

Taylor, J. G. Classical electrodynamics as a distribution theory. II. Proc. Cambridge Philos. Soc. **54** (1958), 258–264.

Continuing an earlier investigation [same Proc. **52** (1956), 119–134; MR **17**, 690] the author studies the definition and conservation property of momentum and energy for classical point charges in interaction with the classical electromagnetic field, giving an entirely rigorous formulation based on L. Schwartz' theory of distributions. The total energy-momentum residing in a space-like surface σ is the "partie finie" of the integral over σ of the stress tensor distribution. The author gives an unambiguous and covariant definition of this "partie finie" and shows that the energy-momentum thus obtained verifies the conservation law. The paper also contains a general definition of vector and tensor distributions, i.e., sets of distributions transforming under the Lorentz group as the components of a vector or a tensor. Many quantities appearing in the earlier and present papers are vector or tensor distributions in this sense. *L. Van Hove* (Utrecht)

4408:

Keller, Joseph B. Propagation of a magnetic field into a superconductor. Phys. Rev. (2) **111** (1958), 1497–1499.

The partial differential equation for the time varying magnetic field in a normal conductor is simplified by assuming that $\sigma\dot{\mathbf{H}} \gg \epsilon\ddot{\mathbf{H}}$. The resulting diffusion equation for **H** is solved subject to the boundary conditions that correspond to the switching on of a field greater than the critical field outside a superconducting wire of circular cross section. The solution is then used to calculate the velocity of the superconducting boundary as the field penetrates the wire, and the time required for the entire wire to reach the normal conducting state. It is shown that the approximation leads to lower bounds for these quantities. Upper bounds are found by solving the analogous problem for a plane superconducting slab. Numerical results are presented graphically. *R. D. Kodis* (Providence, R.I.)

4409:

Heins, Albert E. The Green's function for periodic structures in diffraction theory with an application to parallel plate media. I. J. Math. Mech. **6** (1957), 401–426.

In this paper the author applies the Wiener-Hopf method to solve the problem of diffraction of scalar waves by a semi-infinite set of equidistant parallel plates staggered at a given angle α, for an oblique incident wave making an angle θ with respect to the normal to the direction of the parallel plates. Here the boundary conditions are of the Dirichlet type. This problem is a two-dimensional problem. The Green's function for such a structure must possess periodic properties. Using the boundary condition and the above property of the structure, the author has derived an integral equation of the Wiener-Hopf kind. The solution of this equation is obtained in the usual way by using complex Fourier transforms, first rewriting this equation in a slightly different form in order to apply the Fourier bilateral transformation, this being necessary because of the presence of terms which remain bounded at large distances in the free space part of the structure. The transformed functions are expressed by infinite products involving the separation distance of the plates, b, the wave number k, the order of the wave and the staggering angle α. The reflexion and transmission coefficients (specular reflexion, higher order reflexion, etc.), are calculated for excitation outside the structure and also within the structure (duct propagation). In the latter case mode propagation takes place (reflected and transmitted waves of certain frequencies) and there are also a series of transmitted plane waves in the free space. The effect of the higher modes of the diffracted wave in free space, which result in a mode plane wave transmitted in the duct, the corresponding transmission coefficient due to such types of excitation, and its relation with the transmission coefficient due to the plane waves in free space, are not considered. *N. Chako* (Flushing, N.Y.)

4410:

Heins, Albert E. The Green's function for periodic structures in diffraction theory with an application to parallel plate media. II. J. Math. Mech. **6** (1957), 629–639.

In this part of the paper the author considers the same problem which has been analyzed in part I [see #4409 above], with the Dirichlet boundary condition now being replaced by the Neumann boundary condition. Whereas in the first part the author limited his analysis to wave lengths satisfying the relation $\pi < kb < 2\pi$, the Neumann condition allows, in the parallel plate region, a mode (plane wave) for wave lengths satisfying the relation $0 \leqq kb \leqq \pi$. This problem is also different from that discussed in part I in another respect, i.e., the kernel of the integral equation has a non-integrable singularity. To avoid this difficulty, the author replaces the differential operation under the integral sign by a differential operator acting outside the integral. This procedure is commonly used in such cases by applying the properties of the Green's function and the differential equation satisfied by this function. The resulting integral equation is of the same form as that derived in part I, with two additional plane wave terms containing two arbitrary constants. The solution of this equation by means of Fourier transforms is carried out in the same manner as in part I, with the difference that the coefficients relating the transform functions satisfy two linear relations for the case of specular reflection and $n+2$ relations for the n diffracted waves; this is because of the discontinuity (electromagnetic case) of the z-component of the magnetic field on the plates and the singular behavior of its Fourier transform when its argument approaches infinity. By determining the far field, expressions are given for the transmission and reflexion coefficients for propagated

diffracted waves in the duct and free space and for specularly reflected waves. *N. Chako* (Flushing, N.Y.)

4411:

Jones, D. S. A new method for calculating scattering with particular reference to the circular disc. Comm. Pure Appl. Math. **9** (1956), 713–746.

The problem of finding exact solutions of the diffraction of acoustic or electromagnetic waves by obstacles leads to great mathematical difficulties, except in a few cases where the obstacles are of the simplest geometrical form. In practice, even in some of these simple cases, the results are applicable only to the low frequency range. As a result of the mathematical limitations, one looks for approximate solutions to the problem. In most cases one employs Rayleigh's method, i.e., one first finds a solution of the static problem and, by means of successive approximation, then calculates the radiation problem.

In this paper the author has generalized the method of Rayleigh by using as a first step to the radiation problem, not the static solution, but the non-radiating solution. The advantage of this procedure is that the non-radiating solution (standing-waves) satisfies the wave equation whenever such solutions are known. After a brief discussion of the methods which have been used in attacking the diffraction problem for a circular disc or circular aperture and of the new procedure employed in this paper, the author proceeds to derive an integral-differential equation for the boundary value problem $\partial u/\partial z=0$ on the disc, with the edge condition $u=0$ at the rim of the disc. By using cylindrical coordinates this equation is transformed, after performing the angular integration, to an equation of the Fredholm type of the second kind for the case of low frequencies. The solution of the Fredholm equation is obtained for two incident fields; for a plane wave (normal incidence) and for a point source on the axis.

For the case of the plane wave, the solution of the Fredholm equation is carried out by iteration, which yields a rapidly converging power series expansion in the parameter $\alpha=ka$, where $k=2\pi/\lambda$ and a is the radius of the disc. The results agree with the values given by Bouwkamp up to α^7. In addition, terms in α^8 and α^9 appear here for the first time. On the other hand, when the solution of the Fredholm equation is sought by making approximation to the kernel, i.e., by expanding it in a power series in α, one is led to an infinite system of equations in the unknown coefficients of expansion of the unknown function. The transmission coefficient of the aperture has been calculated for α ranging from 0.1 to 3.0. A comparison with Bouwkamp's results shows that up to $\alpha=2$ they are in excellent agreement with Bouwkamp's values.

For the case of a point source located on the axis and at a distance large in comparison to a from the center of the disc, a procedure similar to that for the case of plane waves is used. A power series solution in α is obtained, with terms up to α^9. In addition, the author gives a detailed analysis of the calculation of the upper bound of the transmission coefficient, as well as the error which is introduced by the method of approximation of the kernel. Numerical values of these quantities for α ranging from 0.1 to 2.0 are tabulated, as well as correction coefficients which must be used in computing the scattering coefficient due to a point source from the known scattering coefficients of an incident plane wave. The appendices contain the mathematical steps which are omitted in the text. *N. Chako* (Flushing, N.Y.)

4412:

Hurd, R. A. Scattering from a small anisotropic ellipsoid. Canad. J. Phys. **36** (1958), 1058–1071.

The author uses a method developed by A. F. Stevenson [J. Appl. Phys. **24** (1953), 1134–1142; MR **15**, 843] to obtain power series expansions of the electromagnetic field scattered by a small homogeneous anisotropic ellipsoid. The series, one for E and one for H, are of the form $H=\sum_0^\infty H^{(n)}(ik)^n$ where $k=2\pi/\lambda$ and λ is the incident wave length. The validity of this form is assumed and convergence is expected when the dimensions of the scattering body are small compared to the wave length inside the body. The incident field is a plane wave. By substituting such series in Maxwell's equations one obtains a recursive system of partial differential equations for the coefficients of the series. By substituting also in the standard electromagnetic boundary conditions which hold at the surface of the scatterer, further conditions on the coefficients are obtained. The recursive system and the algebraic conditions derived from the boundary conditions are used to determine the external and internal scattered field to three terms of the series for each. A special method to calculate the far field, also introduced by Stevenson, is applied in the present paper. *M. Kline* (Aachen)

4413:

Hurd, R. A. The magnetic fields of a ferrite ellipsoid. Canad. J. Phys. **36** (1958), 1072–1083.

The method of the preceding paper [#4412 above] is applied to calculate the magnetic field in and near an ellipsoidal shaped homogeneous ferrite substance and created by an incident plane wave. Since the method consists of obtaining successive coefficients of power series, the author obtains the first two terms for a general ellipsoid and a third term for a spheroid with a special incident magnetic field. Approximate values are found for susceptibility, the ratio of internal magnetization produced to the applied magnetic field, and the results are compared with theoretical and experimental results obtained by others. *M. Kline* (Aachen)

4414:

Rohkind, I. I. Non-stationary diffraction of electromagnetic waves. Vestnik Leningrad. Univ. **13** (1958), no. 7, 109–124. (Russian. English summary)

In this paper the three-dimensional problem of non-stationary diffraction of electromagnetic waves is considered.

The determination of the general solution as a integral identity is introduced according to O. A. Ladyjzenskaja's work.

The method of finite differences was used for deduction of this solution, its existence and uniqueness proved and some differential properties investigated. *Author's summary*

4415:

Van Bladel, Jean. Normal modes methods for boundary-excited wave guides. Z. Angew. Math. Phys. **9a** (1958), 193–202.

The author finds the normal mode expansion of the electromagnetic field in a waveguide excited by an oscillating electric field spanning an orifice in the wall. He does not touch upon the pertinent problem of slow convergence of such expansions. *C. H. Papas* (Pasadena, Calif.)

4416:

Cauer, Wilhelm. **Synthesis of linear communication networks. Vols. 1, 2. 2nd ed., edited by Wilhelm Klein and Franz M. Pelz. Translated by G. E. Knausenberger and J. N. Warfield.** McGraw-Hill Electrical and Electronic Engineering Series. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. xxxvi+866 pp. $19.50.

English translation of "Theorie der linearen Wechselstromschaltungen", second edition, reviewed in MR **16**, 978.

4417:

Okada, Satio; Onodera, Rikio; and Ōrui, Hiroshi. Topological determination of two-port parameters. Bull. Yamagata Univ. (Nat. Sci.) **4** (1958), 333–344.

Without algebraic manipulations, 2-port parameters of electrical networks can be obtained from the network geometry; this is known as topological determination of network functions. This paper tabulates formulas for these parameters with tensor notations. The formulas can be used to analyze networks consisting not only of passive elements, but also of mutual couplings and active elements. Furthermore, the 2-port parameters obtained by the use of these formulas are functions of network elements weighted by either admittance or impedance. Examples show the use of these formulas to analyze networks containing transistors.

W. Mayeda (Yorktown Heights, N.Y.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 4402.

4418:

Buchdahl, H. A. A formal treatment of the consequences of the second law of thermodynamics in Carathéodory's formulation. Z. Physik **152** (1958), 425–439.

The author's aim is to state the content of the second law of thermodynamics in the form given it by C. Carathéodory [Math. Ann. **67** (1909), 355–386] without the explicit use of Carathéodory's theorem on integrating factors of Pfaffian forms. The starting point is Carathéodory's principle: In the neighbourhood of any arbitrary state $\mathfrak{S}$ of an adiabatically enclosed system Σ_0 there are states $\mathfrak{S}'$ inaccessible from $\mathfrak{S}$.

This principle is used to provide a partial ordering of the states of a thermodynamic system. In an adiabatically isolated system Σ_0 the equilibrium states are ordered by the symbols $<$, $>$, $=$, $\geqq$, by the following definitions. If $\mathfrak{S}$ and $\mathfrak{S}'$ are two states of Σ_0 then: (a) $\mathfrak{S}'<\mathfrak{S}$ means that $\mathfrak{S}'$ is inaccessible from $\mathfrak{S}$ by any process, reversible or irreversible, which does not violate the adiabatic condition; (b) $\mathfrak{S}>\mathfrak{S}'$ means that $\mathfrak{S}$ is accessible from $\mathfrak{S}'$, but not conversely; (c) $\mathfrak{S}=\mathfrak{S}'$ means that $\mathfrak{S}$ and $\mathfrak{S}'$ are mutually accessible; and (d) $\mathfrak{S}\geqq\mathfrak{S}'$ means that $\mathfrak{S}$ is accessible from $\mathfrak{S}'$ and the converse may also hold. Carathéodory's principle is extended to include the statement that if $\mathfrak{S}'<\mathfrak{S}$, then necessarily $\mathfrak{S}>\mathfrak{S}'$. This assumption eliminates the possible existence of pairs of states which are mutually inaccessible, and so physically unrelated.

To relate these definitions to the second law of thermodynamics it is assumed that the ordering symbols defined can be replaced by similar symbols which order the numerical values of a real number s. This number, or some monotonic function of it, is destined to play the role of the entropy of the system.

To achieve a complete definition of entropy it is necessary to assume that the set of ordering variables, as determined for all possible adiabatic conditions Σ_0 of a given thermodynamic system Σ, can be organized in the form of a single numerical function of the configuration coordinates and the temperature, such that its values are associated in a unique manner with the states of Σ. The author interprets this condition as a matter of pure empiricism by which the empirical entropy function is established. From this point on the argument follows fairly standard lines in establishing the meaning to be attributed to absolute temperature and absolute entropy.

{It seems to the reviewer that the author understates the problem of establishing the empirical entropy function. The consideration of the possible adiabatically closed conditions of a thermodynamic system establishes at most a class of partial orderings among its complete set of states, each such partial ordering referring to a particular adiabatic condition. To extend these partial orderings to a full ordering of all states is not a trivial step. It is at this point that the usual treatment depends heavily on the properties of the Pfaffian forms giving the description of reversible changes of state. While the author's insistence on the importance of the ordering principle is much to the point, it may be doubted whether it is a completely satisfactory substitute for the customary method of introduction of the entropy concept.}

E. L. Hill (Minneapolis, Minn.)

4419:

Krüger, Friedrich Wolfgang. Die Grundlagen der Thermodynamik. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg **1** (1957), 111–120.

The author presents a treatment of the axiomatic foundations of phenomenological thermodynamics after the general manner of Carathéodory but with a different formulation of the Second Law, in order to circumvent the criticism levelled by Planck [S.-B. Preuss Akad. Wiss. **31** (1926), 453–463] against Carathéodory's original method. The Second Law is stated in the following form: It is impossible to construct a periodic machine which does work whilst merely absorbing a certain quantity of heat from a reservoir. Carathéodory's Theorem can then still be applied by first deducing the existence of states in the neighbourhood of any given state inaccessible from it along (quasi-static) adiabatic paths.

H. A. Buchdahl (Princeton, N.J.)

4420:

Roberts, Leonard. On the melting of a semi-infinite body of ice placed in a hot stream of air. J. Fluid Mech. **4** (1958), 505–528.

"A simple mathematical model is proposed to describe the steady melting of a body of ice which presents a plane surface transverse to a stream of hot air; the temperature of the air is such that vaporization does not occur.

The analysis takes into account the convection of heat away from the surface by the water released in melting and the results show that the rate of transfer of heat to the body and thus the rate of melting, is reduced by as much as 46% by this convection.

Simple approximate expressions are obtained for the rate of melting, the thickness of the water layer, and the thickness of the thermal boundary layer in the ice, in terms of a basic parameter S which can be calculated in terms of known quantities. These results are compared with those obtained by a separate Pohlhausen calculation and are found to be in good agreement.

It is also shown that there exists a thermal boundary

layer, in the body, of thickness much greater than that of the boundary layer in the air, in which the temperature changes rapidly from its value at the melting surface to its value in the far interior." (Author's summary)

C. C. Lin (Cambridge, Mass.)

4421:

Kapur, J. N. The solution of the equations of internal ballistics for power law of burning. Proc. Nat. Inst. Sci. India. Part A. **24** (1958), 15–30.

Vieille (1893) assumed the burning law for a propellent in a closed chamber to be proportional to a constant power of the pressure. The pressure index lies between 0.9 and 1.0 for high power guns, but between 0.3 and 0.8 in rocket motors. For convenience in solving the differential equations, many writers have been content to take the burning rate proportional to the pressure. Chemmow (1928, 1951) reduced the problem of internal ballistics to the solution of an ordinary nonlinear differential equation of order three in the general case, but of order two for tubular grains. This author reviews the entire problem and succeeds, by a suitable change of variables, in reducing the general case to order two. Three methods of integration are proposed under convenient simplifying assumptions. A series solution is given for the case of tubular grains. *A. A. Bennett* (Carbondale, Ill.)

4422:

Sodha, M. S. On internal ballistics of liquid fuel rockets. Appl. Sci. Res. A. **7** (1958), 421–428.

The author obtains the equations for steady state combustion of liquid fuel in droplets, under the following simplifying assumptions: The droplets vaporize completely in the constant volume chamber — none leave the chamber; the chemicals react irreversibly and only in the gaseous state; the gases and vapors mix as soon as formed; the pressure (and for later simplifications also the temperature) remains constant and uniform throughout the chamber. An implicit solution independent of atomization and vaporization laws is obtained under such assumptions. An approximation is found in terms of Bessel and Neumann functions. An approximate transient solution starting when the supply of fuel and oxidizer has been exhausted is given, thus completely solving the problem proposed. *A. A. Bennett* (Carbondale, Ill.)

4423:

Kapur, J. N. Lagrange's ballistic problem for unorthodox [H/L, R.C.L.] guns and solid-fuel rockets. Proc. Nat. Inst. Sci. India. Part A. **24** (1958), 31–39.

The major hydrodynamic problem of internal ballistics is to find for all times and at all points between the breech and the base of the moving shot, the distribution of pressure, density and gas velocity. Lagrange (1793) adopted the hypothesis that the velocity of the gas is at any instant proportional to the distance from the breech. The author reviews this hypothesis in view of the special characteristics of unorthodox guns and of rockets. He concludes that in both cases the distribution of pressure is parabolic. In listing the simplifying assumptions adopted, this author ignores the possibility of travelling waves reflected from the base of the projectile, and of inertia and nonuniform distribution of unburnt charge.

A. A. Bennett (Carbondale, Ill.)

QUANTUM MECHANICS

See also 4026, 4376, 4377, 4441.

4424:

Faddeev, L. D. On the relation between S-matrix and potential for the one-dimensional Schrödinger operator. Dokl. Akad. Nauk SSSR **121** (1958), 63–66. (Russian)

4425:

Matsumoto, Tokuji. Note on the relativistic wave equation. J. Fac. Sci. Hokkaido Univ. Ser. II. **4** (1955), 411–419.

In this paper, the author studies the structure of relativistic wave equations in which the wave function transforms as the direct sum of two Dirac spinors, with special attention to irreducible equations and to distinct mass levels. Certain coupled systems, characterized by suitable Lagrangians, are singled out and it is claimed that the criterion of irreducibility requires that there be only a single mass level. However, as far as this reviewer can verify, the criterion of irreducibility is arbitrary and a similarity transformation always exists which separates the Lagrangian into the sum of the Lagrangians of two uncoupled Dirac fields with generally unequal masses.

E. C. G. Sudarshan (Cambridge, Mass.)

4426:

✱Weizel, Walter. Lehrbuch der theoretischen Physik. 2. verbesserte Aufl. 2. Bd: Struktur der Materie. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. xv+989 pp. DM 88.00.

This is a contribution of high merit to the pedagogical literature of theoretical physics at the level of graduate study. It presents a unified exposition of the subject matter centering around quantum mechanics and its applications to atomic, molecular, and nuclear structure theory. There are many extensions to the theory of gases, electronic phenomena (electron and ion optics, gas discharges), crystal structure, and metals, with short sections on semiconductors and the liquid state. The two sections on quantum and field theory constitute one of the best text-books on the subject known to the reviewer.

The author has written in an encyclopedic manner, and the compact style will require the reader's careful attention. The material is condensed, but well selected and presented. Each sub-section is preceded by a brief statement of its contents.

The mathematical discussions are treated, on the whole, at about the level at which they are accepted in the current physical literature. The Schrödinger equation and its statistical interpretation are introduced almost axiomatically, and no attempt is made to deal with the subtleties of operator theory. Matrix mechanics is presented formally, and the complexities of continuous spectra are set aside with the briefest mention. The mathematically sophisticated reader must, therefore, look elsewhere for the deeper treatment of the subject.

This work is recommended to the teacher seeking source material, to the student for home study, and to the general reader who wishes a comprehensive review of modern theoretical physics.

{The reviewer finds it regrettable that MKS units have been employed in this book. This concession to the practical needs of the engineer seems out of place in a purely theoretical discussion}.

E. L. Hill (Minneapolis, Minn.)

4427:

Matthews, P. T.; and Salam, Abdus. Relativistic field theory of unstable particles. Phys. Rev. (2) **112** (1958), 283–287.

Recent exploitation of exact properties of quantum field theory has depended upon a construction for one-particle states based upon the asymptotic condition. For unstable particles no such contruction exists. It is questionable whether consistent calculations can be made beyond the lowest order in perturbation theory, and there appears to be no exact theory to which this is supposed to be an approximation.

The paper under review attempts to provide the beginnings of such an exact theory. A description of unstable particles in terms of poles in the complex energy plane is rejected. Rather, the starting point is a spectral function, ρ, closely related to that defined by Lehmann. An unstable state corresponds to a finite peak in ρ, and its lifetime is a measure of the width of that peak.

The theory is to be developed in ensuing papers.

J. C. Taylor (London)

4428:

Fischer, Jan. Equations for the Green functions in quantum electrodynamics. Czechoslovak J. Phys. **8** (1958), 379–389. (Russian summary)

Some properties of Green's functions for a system of electrons and photons are discussed. The paper contains a number of complicated formulae, but it is not clear whether they are of any use at all from a practical point of view. *S. N. Gupta* (Detroit, Mich.)

4429:

Rayski, Jerzy. An attempt to geometrize mesoelectrodynamics. Acta Phys. Polon. **17** (1958), 187–198.

The author introduces a six-dimensional space by combining the four-dimensional Minkowski space and the two-dimensional isotopic spin space. He then shows that several well-known properties of the electromagnetic and the meson fields can be given a geometrical interpretation in the six-dimensional space. *S. N. Gupta* (Detroit, Mich.)

4430:

Nagy, K. L. Tomonaga's intermediate coupling theory using configuration space methods. Acta Phys. Acad. Sci. Hungar. **9** (1958), 23–48. (Russian summary)

The structure of the nucleon is considered here in an intermediate coupling theory. First the state vector of the real nucleon is written down. Nucleon pair creation is neglected. Then the scalar meson field is considered with and without nucleon recoil. Physical quantities like the mean values of the meson potential, the electric charge density and the energy density are calculated. The many-nucleon problem is also discussed. Finally the pseudoscalar meson field is considered. The paper ties in with work done by Heber [Ann. Physique (6) **16** (1955), 43–51; MR **17**, 566] and by Sachs [Phys. Rev. (2) **87** (1952), 1100–1110]. *M. J. Moravcsik* (Livermore, Calif.)

4431:

Borovikov, V. A. A topological problem connected with questions of quantum electrodynamics. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 113–118. (Russian)

Consider a planar graph P with a finite number of vertices, which are connected among each other by lines of two sorts, electron and photon lines, where two electron lines and one photon line issue from each vertex. Such graphs are important for computing the probabilities, in the form of integrals, of various processes in electrodynamics. The computations associated with P are facilitated if P has parts of various kinds. Thus, if P has a part connected with the rest of P by a single line, then the corresponding integral is 0. This trivial case is set aside. If P is disconnected, then the corresponding integral is the product of two integrals. This case is also set aside. If P has a part A connected with the remaining part of P by exactly two lines (both electron or both photon), then A is called an S-subscheme of P. If P has a part A connected with the remaining part of P by exactly three lines, then A is called a V-subscheme. If P contains an S-subscheme or a V-subscheme, then the integrand corresponding to P is the product of two functions. Hence it is useful to reduce graphs P that contain such subschemes. If P contains an S-subscheme A, let P' be the graph obtained from P by eliminating A and replacing the lines α and β connecting A by a single line γ of the same sort as α and β, with endpoints at the endpoints of α and β. If P contains a V-subscheme A, let P' be the graph obtained from P by shrinking A to a single vertex. Any graph P can be reduced by a repetition of these processes to a graph Q containing no S- or V-subschemes. Theorem: The graph Q is uniquely determined by P. *E. Hewitt* (Seattle, Wash.)

4432:

Borovikov, V. A. A topological problem connected with questions of quantum electrodynamics. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **11** (1957), no. 2, 31–37. (Romanian)

Translation of a Russian paper [reviewed above].

4433:

✱High energy nuclear physics. Proceedings of the Seventh Annual Rochester Conference, April 15-19, 1957. Compiled and edited by G. Ascoli, G. Feldman, L. J. Koester, Jr., R. Newton, W. Riesenfeld, M. Ross, and R. G. Sachs. Interscience Publishers, Inc., New York, 1958. ix+486 pp. $4.50.

Introductory surveys, short reports and discussion are here recorded more or less verbatim. The style is therefore informal and sometimes a little tedious in bulk. The compilation is well produced, with many excellent diagrams.

The mass of the experimental information has since been published formally, and no longer commands great interest. Among the surveys that can still be read with profit are those by Chew on nucleon structure, by Cassels on pion reactions, and by Lee on weak interactions.

At the time of the Conference, dispersion relations had been applied in some detail to pion scattering and photoproduction, and the pion coupling constant had begun to be known accurately (though the Puppi anomaly was unresolved). Analysis of nucleon-nucleon scattering had reached a point where two potentials, due to Marshak and Signell, and Gammel and Thaler, could seriously be proposed.

Gell-Mann, Schwinger and Tiomno had conceived the idea of "global symmetry" in the strong interactions — an idea which is still very much alive.

The lifetime of the π° had been corrected. The Σ° had been found. The predicted behaviour of the $K^\circ-\bar{K}^\circ$ mixture had been partly verified.

Following the discovery that parity is not always a good quantum number, a new picture of the weak interactions had begun to appear, and many beautiful experiments are described in the report. Nevertheless, there

was discomfort that a universal Fermi interaction did not appear to fit the facts. The historian will note that, already at that time, Feynman was theorizing in a way which would lead him, among many others, to question widely accepted experimental conclusions on β-decay. The magnetic moment of the muon had been measured with accuracy.

A discussion involving Schwinger and Källén on the analytic properties of Green functions was the first intimation to many people of the richness, as well as the difficulty, of this type of enquiry. *J. C. Taylor* (London)

4434:

Barut, A. O. A theory of particles of spin one-half. Ann. Physics 5 (1958), 95–105.

The solutions ψ of the Dirac equation may be written in terms of a 2-component ϕ which satisfies a second order wave equation [R. P. Feynman and M. Gell-Mann, Phys. Rev. **109** (1958), 193–198; MR **19**, 813]. The author constructs from this ϕ and its time derivative another 4-component wave function Ψ which satisfies the equation

$$i\frac{\partial\Psi}{\partial t}=H\Psi,$$

where

$$H=(2m)^{-1}[(-i\nabla-e\mathbf{A})^2+\tfrac{1}{2}ei\gamma^\mu\gamma^\nu F_{\mu\nu}]\gamma^0(1+\gamma^5)-m\gamma^0+eA_0.$$

He remarks that for free particles the spin and orbital parts of the angular momentum are readily separable in this representation, and discusses the relation of his formalism to that of Foldy and Wouthuysen [L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78** (1950), 29–36].

P. W. Higgs (London)

4435:

Puzikov, L. D. Scattering of particles of arbitrary spin. Soviet Physics. JETP 34(7) (1958), 655–658 (947–952 of Russian original).

The elastic scattering matrix for particles of arbitrary spin and finite mass is constructed in terms of the irreducible spin tensors and the initial and final wave vectors. The physically measurable quantities like cross sections and polarizations are computed in terms of the invariant (energy-dependent) parameters of the scattering matrix. The problem of reconstructing the scattering matrix from a "complete set" of experiments is discussed. The formalism developed is not applicable to particles of non-zero spin and zero mass; and no mention is made of methods to deal with these singular cases.

E. C. G. Sudarshan (Cambridge, Mass.)

4436:

Ladányi, K. Variational method for the solution of the quantum-mechanical many-body problem. Acta Phys. Acad. Sci. Hungar. 9 (1958), 115–124. (Russian summary)

This paper gives a generalization of the variational method of W. Macke [Phys. Rev. **100** (1955), 992–993] in the Thomas-Fermi approach for the case of a spherically symmetric potential. The method is applied to the A atom, giving results in good agreement with results obtained by other methods. *D. ter Haar* (Oxford)

4437:

Beliaev, S. T. Application of the methods of quantum field theory to a system of bosons. Soviet Phys. JETP 7 (1958), 289–299.

In this paper the author applies the methods of the quantum theory of fields to systems consisting of a large number of bosons and constructs the equation satisfied by the one-particle Green's function. An elegant trick of separating out the operators corresponding to zero-momentum enables the author to apply the technique of Feynman graphs [R. P. Feyman, Phys. Rev. **76** (1949), 749–759] to the analysis of the propagation of particles of non-zero momentum. The solution of the Green's function equation by approximation methods is discussed. Perturbation approximation to the effective potential occurring in this equation leads to the quasi-particle spectrum of excitations of Bogoliubov [Izv. Akad. Nauk SSSR Ser. Fiz. **11** (1947), 77–90; MR **9**, 72]. The demonstration that the condensed phase with a finite fraction of particles of zero-momentum persists even in the presence of interactions is not convincing.

E. C. G. Sudarshan (Cambridge, Mass.)

RELATIVITY

See also 4425.

4438:

Kohler, Max. Invariante Flächen der Elektrodynamik. Z. Physik **150** (1957), 118–122 (1958).

In a previous article [Z. Physik **148** (1957), 443–453; MR **19**, 495] the author has shown that there exist two families of invariant surfaces orthogonal to each other, in a space-time free of charges and currents, which are associated with the electromagnetic field tensor (six-vector) M_{ij} $(i, j=1, 2, 3, 4)$. The vanishing of the current-density vector P_k $(k=1, 2, 3, 4)$ is a sufficient but not necessary condition for the existence of the second family. In this paper the author gives the necessary and sufficient conditions for the existence of the second family of invariant surfaces. In the case of a charge and current free space the proof of the existence of the second family is obtained by introducing a dual tensor to M_{ij}, M_{ij}^+, which satisfies the divergence condition and the Pfaffian equation $M_{ij}^+dx^k=0$ $(i=1, 2, 3, 4)$. For the case where the four-current density vector does not vanish, the necessary and sufficient conditions for the existence of the second family of surfaces is that $M_{ij}^+P^k=0$. A well known special case of these conditions is when one considers stationary magnetic fields,namely $J\cdot B=0$, i.e., the space current density is normal to the magnetic field vector B.

N. Chako (Flushing, N.Y.)

4439:

Bergmann, Peter G.; and Janis, Allen I. Subsidiary conditions in covariant theories. Phys. Rev. (2) **111** (1958), 1191–1200.

This paper is one of a series in which Bergmann and his collaborators are attempting to determine the observables of general relativity, that is, functions (or functionals) of the field variables that are invariant with respect to co-ordinate transformations, and so correspond to physically measurable quantities. The values of such quantities at a time t can be predicted from a sufficient set of data referring to an earlier time t_0, despite the fact that the permissible co-ordinate transformations depend on arbitrary functions of the time.

This particular paper examines the effect of introducing co-ordinate conditions on the formalism established in previous papers [E. Newman and P. G. Bergmann, Rev. Mod. Phys. **29** (1957), 443–449; MR **19**, 1139]. An exact procedure is described for determining the observables in

the presence of co-ordinate conditions, but, unfortunately, this procedure is no easier to follow than when co-ordinate conditions are not assumed. It is, therefore, suggested that the introduction of co-ordinate conditions should be combined with the approximation method of Newman and Bergmann [loc. cit.]. *D. W. Sciama* (London)

4440:

Callaway, Joseph. Klein-Gordon and Dirac equations in general relativity. Phys. Rev. (2) **112** (1958), 290.

"It is shown that, when the gravitational field of a point charge is considered, the Klein-Gordon and Dirac equations for the motion of a charged particle in a Coulomb field do not possess solutions which can be expressed as series of terms proportional to positive integral powers of the gravitational constant." (Author's summary)
P. W. Higgs (London)

4441:

Komar, Arthur. Construction of a complete set of independent observables in the general theory of relativity. Phys. Rev. (2) **111** (1958), 1182–1187.

Utilizing the four scalars of Geheniau and Debever [Helv. Phys. Acta Suppl. **4** (1956), 101–105] which do not vanish if the field equations of general relativity are satisfied in vacuo ("exterior case"), Komar has constructed a complete, though redundant set of observables, i.e., of quantities whose values are independent of any chosen coordinate system and which, together, completely characterize the geometry of a Riemann-Einstein manifold. As far as the reviewer knows, this is the first instance in which a complete set of these observables has been constructed in closed form. They may be viewed, alternatively, as the components of the metric tensor in a uniquely fixed special coordinate system. The paper also discusses the implications of this construction on the Cauchy problem of general relativity and on the problem of quantization. *P. G. Bergmann* (New York, N.Y.)

4442:

Petrov, A. Z. Regular Einstein spaces admitting a transitive group of motions. Kazan. Gos. Univ. Uč. Zap. **112** (1952), no. 10, 27–33. (Russian)

4443:

Petrov, A. Z. Gravitational fields with complex stationary curvatures. Kazan. Gos. Univ. Uč. Zap. **112** (1952), no. 10, 35–47. (Russian)

4444:

Pachner, Jaroslav. Die unitäre Feldtheorie in der Maxwellschen Näherung. Ann. Physik (7) **2** (1958), 36–40.

The author shows that by adding a certain scalar density to the Hamiltonian in Einstein's theory one obtains field equations from which the equations of motion in a weak electromagnetic field follow.
L. Infeld (Warsaw)

ASTRONOMY

4445:

Ueno, Sueo. The probabilistic method for problems of radiative transfer. II. Milne's problem. Astrophys. J. **126** (1957), 413–417.

[For part I, see Contributions Inst. Astrophys. Kyoto Univ., no. **64** (1956)]. This paper shows that a form of the equation of radiative transfer for isotropic scattering in a semi-infinite medium, due to Ambartsumian, can be derived from the Chapman-Kolmogorov equation in the theory of Markov processes by essentially the same procedure that is used to derive the Kolmogorov-Feller equation. A somewhat novel method is given for solving the so-called gray-body problem.
D. Layzer (Cambridge, Mass.)

4446:

Sobolev, V. V. The diffusion of radiation in a medium of finite optical thickness. Astr. Ž. **34** (1957), 336–348. (Russian. English summary)

The ideas developed in this paper are similar to those in a recent paper by Ueno [reviewed above]. The novel point is the introduction of the function $p(\mu;t)$ which gives the probability of emission at a given point t in the medium and at a certain direction specified by the direction cosine μ. If the process is Markovian, p clearly satisfies the integral equation

$$p(\mu;t)=\int_0^1 p(\mu';t-\tau)p(\mu/\mu';\tau)d\mu'$$

where $p(\mu_2/\mu_1;t_2,t_1)$ (with $t_2>t_1$) is the transition probability that given $\mu(t_1)=\mu_1$, one finds μ in the range $(\mu_2,\mu_2+d\mu_2)$ at depth t_2. From this one can derive

$$(*)\quad p(\mu;t)=p(\mu;0)\Big[e^{-t/\mu}+\tfrac{1}{2}\int_0^t ds\int_0^1 e^{-(t-s)/\mu}p(\mu';s)\frac{d\mu'}{\mu'}\Big].$$

This equation forms the basis of the present paper. The author shows how, from (*), the integral equations for the transmission and the scattering functions that follow from certain principles of invariance [S. Chandrasekhar, Radiative transfer, Clarendon Press, Oxford 1950; MR **13**, 136] can be deduced. Methods for getting asymptotic expansions for these functions for large optical thicknesses are also discussed.
S. Chandrasekhar (Williams Bay, Wis.)

GEOPHYSICS

4447:

Herzenberg, A. Geomagnetic dynamos. Philos. Trans. Roy. Soc. London. Ser. A. **250** (1958), 543–583.

The author considers a sphere filled with conducting matter within which there are two smaller non-overlapping spheres. The matter within each small sphere rotates as a rigid body at a constant angular velocity about a fixed axis, whereas the matter without the small spheres is stationary. He shows that the arrangement acts as a dynamo in which each of the smaller spheres rotates in the magnetic field arising from induction in the other. In this way the two spheres "feed" each other and produce a magnetic field that extends outside the main sphere. *C. H. Papas* (Pasadena, Calif.)

OPERATIONS RESEARCH AND ECONOMETRICS

See also 4234, 4320.

4448:

Friedman, Lawrence. Game-theory models in the allocation of advertising expenditures. Operations Res. **6** (1958), 699–709.

The author isolates the competitive aspect of adver-

tising, and formalizes it as a two-person, zero-sum game. Using well-known methods and results, he obtains interesting information on optimum allocation strategy under various assumptions. *J. H. Blau* (Yellow Springs, Ohio)

4449:

Flood, Merrill M. Some experimental games. Management Sci. 5 (1958), 5–26.

This paper reports the results of six experiments and analyses performed to explore the applicability of the non-constant sum case of the theories of von Neumann-Morgenstern, and others, to the actual behavior of people playing games or involved in bargaining situations. The paper suggests directions in which the theory of games might be modified and extended to improve its applicability and usefulness. A 'split-the-difference principle' is suggested to augment the usual theory, so as to specify the exact amount of payments to be made in an ordinary two-person bargaining situation such as the sale of a used car. The application of this principle seems satisfactory in the experiments. One experiment suggests that, in a sequence of trials in the same game situation, people tend to start near an equilibrium point and then try to find a better equilibrium, if there is one. The experiments show examples of non-optimal behavior of the bargainers when the judgment necessary to estimate the relevant payoff is obscure. A fair division of five parcels of objects among five players when each player attaches different values to the parcels is outlined and computed, and the effect of coalitions is discussed. (From the author's summary.)

M. Dresher (Pacific Palisades, Calif.)

INFORMATION AND COMMUNICATION THEORY

See also 4333, 4357.

4450:

Kolmogorov, A. N. The theory of transmission of information. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 8 (1958), 113–142. (Hungarian)

Translated by P. Medgyessy, from the author's paper presented before the Session of the Academy of Science of the U.S.S.R. on the scientific problems of automatic production, 15–20 October 1956, Plenary Session, pp. 66–99, Izdat. Akad. Nauk SSSR, Moscow, 1956.

E. Lukacs (Washington, D.C.)

4451:

Cuzzer, A. Limiti e possibilità del concetto di informazione. Period. Mat. (4) 36 (1958), 41–47.

CONTROL SYSTEMS

See also 3785, 4039.

4452:

Rieger, Ladislav. On the theory of the neural nets. Apl. Mat. 3 (1958), 243–274. (Czech. Russian and English summaries)

This is a survey of the theory of neural nets, based primarily on the work of Russian and American authors. Topics treated include relay and electronic representations of neurons, the representation of events by finite automata, and problems of synthesis.

V. E. Beneš (Murray Hill, N.J.)

4453:

Krasovskii, N. N. On a problem of optimal control. Prikl. Mat. Meh. 21 (1957), 670–677. (Russian)

Consider a control system characterized by the vector difference equation

$$x[(k+1)h]-x(kh)=[Ax(kh)+Bu(kh)+f(kh)]h \qquad (k=0, 1, 2, \cdots),$$

where $x(kh)$ is the state vector at time kh, A and B are constant matrices, $u(kh)$ is the control vector, $f(kh)$ is a given input vector and h is the sampling interval. The optimal control problem consists of finding within a class of allowable control vectors a sequence $u(o), \cdots, u(Nh)$ which takes the system from a specified state at $t=0$ to a specified state at $t=(N+1)h$ in the shortest possible time. The author shows that for the case where the control vectors are constrained by $\max_k u(kh)\leq L=$given constant, the problem of optimal control reduces to a variational problem in the theory of moments which was considered by Krein [On some questions in the theory of moments, Gosudarstv. Nauč.-Tehn. Izdat. Ukrainy, Kharkov, 1938]. For the limiting case $h\to 0$, the results given in this paper reduce to results presented in a related paper dealing with continuous parameter systems [Avtomat. i Telemeh. 18 (1957), 960–970; MR 20 #803]. See also Boltyanskii, Gamkrelidze and Pontryagin [Dokl. Akad. Nauk SSSR 110 (1956), 7–10; MR 18, 859].

L. A. Zadeh (New York, N.Y.)

4454:

Greniewski, M. $(m+n)$-element algebras and their applications to relay-contact systems. Zastos. Mat. 4 (1958), 142–168. (Polish. Russian and English summaries)

The author develops a theory of universal functions realizing the mappings upon an m-element set A of the Cartesian product of the pth Cartesian power of a set A and the qth Cartesian power of a set B (the set B contains n elements). Substituting suitable values for the parameters of the universal functions we obtain each of the mappings. The author then investigates the properties of universal functions and introduces the algebraic equivalents of delay and feed-back. He deals with the particular cases of the above algebra, investigating the algebra in which $m=2$ and $n=3$ and that in which $m=3$ and $n=2$. He discusses applications of those algebras, e.g., to relay-contact systems. *From the author's summary*

HISTORY AND BIOGRAPHY

4455:

Gentile, Giovanni. Sul problema di Delo. Archimede 10 (1958), 118–120.

The problem of Delos, the duplication of the cube, and its generalizations using curves of the second order (Menaechmus, Cartesius, Sluse, Grégoire, Longchamps) can be represented and solved in a unified form by means of the equation $m(x^2-ay)+n(y^2-bx)+p(xy-ab)=0$, representing a system of ∞^2 conics, which for special values of the constants m, n, p brings into evidence the different known classical solutions. *S. R. Struik* (Cambridge, Mass.)

4456:

Yuškevič, A. P. On the achievements of Chinese scholars in the field of mathematics. Advancement in Math. 2 (1956), 256–278. (Chinese)

A translation from the Russian original in Istor.-Mat. Issled. 8 (1955), 539–572 [MR 17, 1].

4457:

Van, Š. I. Bibliography on the achievements of Chinese mathematicians. Advancement in Math. 4 (1958), 306–312. (Chinese)

4458:

Bicadze, A. V. Mathematics during 40 years in the USSR (brief survey). Advancement in Math. 4 (1958), 583–585. (Chinese)

4459:

Čen', Czyan-gun. Notes on Soviet mathematicians and Soviet mathematics. Advancement in Math. 4 (1958), 586–590. (Chinese)

4460:

Biernacki, M. Sur les travaux de la théorie de fonctions en Pologne. Ann. Acad. Sci. Fenn. Ser. A. I, no. **251/1** (1958), 11 pp.

An account of the work of Polish mathematicians since the war with special emphasis on the theory of functions.

4461:

Walusinski, Gilbert. Au pays de Clairaut et de Bourbaki. Enseignement Math. (2) 3 (1957), 289–297.

The author takes Bourbaki as the exponent of the axiomatic direction and Clairaut as the exponent of the (in some respects opposite) practical direction in mathematics, discusses the aims of the French Teachers' Organisation in changing the program and teaching methods, and reviews the results obtained in trying to obtain an equilibrium between the two methods of approach. *E. M. Bruins* (Amsterdam)

4462:

Scriba, Christoph J. James Gregory's frühe Schriften zur Infinitesimalrechnung. Mitt. Math. Sem. Giessen no. 55 (1957), 80 pp.

This monograph presents examples of the work of a brilliant but short-lived younger contemporary of Newton and Leibniz, whose accomplishments remained largely unknown until 1939. The author displays Gregory's results in modern notation, but he also takes care to lead the reader through Gregory's strongly geometric and verbal formulations and methods.

In the *Vera Quadratura* (publ. 1667) is found an elegant method for computing the area of a sector of any central conic. This is applied to the circle and the hyperbola to obtain fifteen-place and twenty-three place approximations to π and ln 10, respectively. Gregory goes on to attack the general problem of computing a table of logarithms.

The *Geometria* (publ. 1668) was written as a textbook in which Gregory organized all that he had learned of the new infinitesimal methods during a stay in Padua. Of particular interest is his method of rectifying curves.

The *Exercitationes Geometricae* are small treatises on topics including various series expansions, the integral of the logarithmic function, and so on. *E. S. Kennedy* (Beirut)

4463:

Brun, Viggo; and Jessen, Børge. A letter by Niels Henrick Abel from his youth. Nordisk Mat. Tidskr. 6 (1958), 21–24, 56. (Norwegian. English summary)

The letter contains nothing of mathematical interest. *From the authors' summary*

4464:

Biermann, Kurt-R.; und Brun, Viggo. Eine Notiz N. H. Abels für A. L. Crelle auf einem Manuskript Otto Auberts. Nordisk Mat. Tidskr. 6 (1958), 84–86, 96.

4465:

Jones, Harold Spencer. Edmond Halley and his work. Proc. Roy. Inst. Great Britain (1956), 20 pp.

A non-technical biographical sketch. There is no bibliography.

4466:

Freudenthal, Hans. Zur Geschichte der Grundlagen der Geometrie. Zugleich eine Besprechung der 8. Aufl. von Hilberts "Grundlagen der Geometrie". Nieuw Arch. Wisk. (3) 5 (1957), 105–142.

On the occasion of the eighth edition of Hilbert's "Grundlagen der Geometrie" [Teubner, Stuttgart, 1956; MR 18, 227], prepared by P. Bernays, the author appropriately offers an historical perspective of this fundamental work of our times. The contributions of antiquity to the analysis of unsolved problems (Euclid, Eudoxus) successfully attacked with modern means, the 19th century contributions up to Hilbert in the field of axiomatics (Riemann, Lie, Klein, Pasch, Veronese, Pieri, Russell, etc.) and those expressed progressively in the different editions of the book (Padoa, Dehn, Hessenberg, Poincaré, etc.) are here discussed, especially with respect to questions of independence of axioms, continuity, theory of proportions, etc. The author also gives an account of other modern developments (van der Waerden, R. Moufang, D. van Dantzig, E. Artin, etc.). The author agrees with Bernays' decision to omit those chapters of Hilbert's book which are not purely geometric and takes issue with some detailed aspects of this eighth edition. {Realizing the significance of the book, it is amazing that it should be available in English only in a translation of an early edition, which Hilbert himself later modified repeatedly.} *S. R. Struik* (Cambridge, Mass.)

4467:

Martin, D. Sir Edmund Whittaker, F. R. S. Proc. Edinburgh Math. Soc. 11 (1958), 1–9. (1 plate)

A general biographical sketch with one photograph. There is no bibliography.

4468:

Rankin, R. A. Sir Edmund Whittaker's work on automorphic functions. Proc. Edinburgh Math. Soc. 11 (1958), 25–30.

This paper consists of an examination of the work of E. T. Whittaker on automorphic functions together with a summary of the work on automorphic functions of other mathematicians (Mursi, J. M. Whittaker, Dalzell, Dhar, Hodgkinson) which was stimulated by a conjecture made by E. T. Whittaker in 1929. *M. H. Heins* (Urbana, Ill.)

4469:
Amitsur, Shimshon Avraham. The scientific work of Prof. Jakob Levitzki. Riveon Lematematika **11** (1957), 1–6. (Hebrew) (1 plate)
A short scientific biography with a bibliography of 51 entries.

4470:
Brun, Viggo. Carl Störmer in memoriam. Acta Math. **100** (1958), pp. i–vii.
A brief biography and a bibliography of 33 entries.

GENERAL

4471:
✶Alexandroff, P. S.; Markuschewitsch, A. I.; und Chintschin, A. J. (Redakteure) Enzyklopädie der Elementarmathematik. Bd. 3. Analysis. Hochschulbücher für Mathematik. Bd. 9. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958. ix+536 pp. DM 30.00.
There are 3 sections; the first by W. L. Gontscharow, Elementare Funktionen einer reellen Veränderlichen, Grenzwerte von Folgen und Funktionen. Der allgemeine Funktionsbegriff; the second by I. P. Natanson, Ableitungen, Integrale Reihen; the third, again by W. L. Gontscharow, Elementare Funktionen einer komplexen Veränderlichen. For the first two volumes see MR **17**, 445 and **18**, 183.

4472:
✶Behnke, H.; Fladt, K.; und Süss, W. (Herausgeber). Grundzüge der Mathematik: für Lehrer an Gymnasien sowie für Mathematiker in Industrie und Wirtschaft. Auf Veranlassung des Deutschen Unterausschusses der Internationalen Mathematischen Unterrichtskommission. Bd. 1. Grundlagen der Mathematik: Arithmetik und Algebra. Göttingen, Vandenhoeck & Ruprecht, 1958. xii+558 pp. (1 insert) DM 50.00.
This volume is the first of four that are planned; the titles of the other three are to be: Geometry; Analysis; and Practical methods and applications of mathematics. The total number of authors will be more than 100, of whom 31 take part in the present volume. Each article has two authors, one of them a university professor, the other a teacher in a gymnasium. There are two parts: Grundlagen der Mathematik, and Arithmetik und Algebra. The second part has ten chapters, with the titles: Aufbau des Systems der reellen Zahlen (mit Anhang: Ordinalzahlen), Gruppen, Lineare Algebra, Polynome, Ringe und Ideale, Zahlentheorie, Algebraische Körperweiterung, Komplexe Zahlen und Quaternionen, Verbände, Einige Grundbegriffe der Strukturtheorie.

4473:
✶Abellanas Cebollero, Pedro. Elementos de matematica. [Elements of mathematics.] Madrid, 1958. viii+376 pp. 300 pesetas.
An introduction to real and complex numbers, matrices, vectors, euclidean geometry, sequences and series, real and complex functions, and differential and integral calculus. The approach is modern; for example, on page 2 the integers are stated to form an additive group.

4474:
✶Burger, D. Bolland. [Sphere-land.] C. Blommendaal N.V., Hague, 1957. 120 pp.
This novel, with the subtitle "A novel of curved spaces and expanding universe by a Hexagon, with a glimpse into Flatland, a fantasy on the fourth dimension by a Square", is a continuation of E. A. Abbott's "Flatland" (1884). A grandson of Abbott's Square, writing seventy years later, gives details about the physical constitution of Flatland in an entertaining way and tells how certain clever people in Flatland begin to realize that their world is really a section of a large spherical surface, which itself is expanding in three space. A very human story.

D. J. Struik (Cambridge, Mass.)

titles of the latter there are to be: Geometry, Analysis, and Practice; methods and applications of mathematics. The total number of authors will be more than 100, of whom 21 take part in the present volume. Each article has two authors, one of them a university professor, the other a teacher in a gymnasium. There are two parts: Grundlagen der Mathematik, and Arithmetik und Algebra. The second part has ten chapters, with the titles: [illegible] Kombinatorik, [illegible] Ringe und Ideale, Zahlentheorie, [illegible] Komplexe Zahlen und Quaternionen, [illegible] Einige Grundbegriffe der [illegible].

4473:

♦Abellanas Cebollero, Pedro. Elementos de matemática. [Elements of mathematics.] Madrid, 1956. [illegible] pp.

An introduction to real and complex numbers, [illegible] vectors, euclidean geometry, [illegible] and complex functions, and differential and integral calculus. The approach is modern [illegible]

4474:

♦Burger, D. Bolland. [Sphere-land.] [illegible] N.V., Hague, 1957. 135 pp.

The novel [illegible] describes [illegible] and expanding universe [illegible] Flatland [illegible]

4475:

Amitsur, Shimshon Avraham. The scientific work of Prof. Jakob Levitzki. Riveon Lematematika 11 (1957), 1–6. (Hebrew) [illegible]

A short scientific biography with a bibliography of 21 items.

4476:

[illegible] Carl Wilhelm Borchardt. Acta Math. 100 (1958), [illegible]

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